Optimum Power Allocation in Sensor Networks for Passive Radar Applications

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***Abstract*—In the present work, we investigate the power allocation problem in distributed sensor networks that are used for passive radar applications. The signal emitted by a target is observed by the sensor nodes independently. Since these local observations are noisy and are thus unreliable, they are fused together as a single reliable observation at a remotely located fusion center in order to increase the overall system performance. The fusion center uses the best linear unbiased estimator in order to estimate the present target signal accurately. By using the proposed system model, fusion rule and objective function we are able to optimize the power allocation analytically and can hence present a closed-form solution to the power allocation problem. Since the power allocation problem can be subject to different power constraints, three different cases of power constraints are discussed and compared with each other. Furthermore, we demonstrate that all considered constraints lead to signomial optimization problems which are in general quite hard to solve. The main applications of the proposed results are issues concerning the sensor selection and energy efficiency in passive sensor networks.**

***Index Terms*—Analytical power allocation, energy-efficient optimization, distributed radar, network resource management, information fusion.**

I. Introduction

IN this publication, we consider a sensor network where each of K nodes individually and independently receives a signal from a jointly observed target source. The type of the source and its signal are assumed to be abstract; only the quadratic mean of the radiated target signal needs to be known. The particular information about the target signal at each sensor node (SN) is sent to a fusion center, which combines the local observations into a single quantity in order to increase the system performance. This setup is illustrated in Fig. 1 whose technical components will be specified in detail later. Both the sensing and the communication channels are subject to additive noise. Moreover, we assume that the SNs have only limited sum-power available for communication and that each SN is in addition limited in its transmission power-range. A potential application of our approach is passive multiple-radar sensing, where an unknown target signal shall be estimated, detected or classified. Instead of using a complex single-radar system, this task is carried out by a network of cheap and energy-efficient SNs. To achieve comparable system performance, a fusion center combines a multitude of local observations into a single reliable quantity.

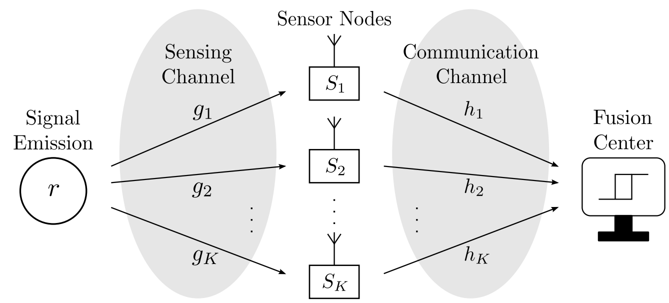

Fig. 1. Abstract representation of the distributed sensor network.

Distributed passive multiple-radar systems (DPMRSs) have worthwhile applications nowadays. Physicists use this type of radar to detect or to determine specific characteristics of particles, for example, in the neutrino telescope '*IceCube Neutrino Observatory*' at the Amundsen-Scott South Pole Station [2] in Antarctica, where a network with over 5000 nodes is implemented. They also use such radars for radio astronomy to study celestial objects, for instance in the '*Karl G. Jansky Very Large Array*' of the National Radio Astronomy Observatory [3] in Socorro County, New Mexico. Many other applications of DPMRSs are military [4] and some few are also for civil uses [5]. Because of the significance of DPMRSs it is important to investigate the power allocation within the sensor network in order to improve the radar accuracy while the overall energy consumption is kept constant [6]. For comparing the performance of different power allocation methods in energy-efficient systems, the problem thus arises how to allocate a given sum-power to the SNs for transmitting the local observations to a fusion center. The problem of finding an optimum power allocation for a distributed radar system and a closed-form of the objective function is extremely hard, and it is even harder to determine optimal points under certain constraints. The main difficulty is associated with finding an explicit representation of the objective function as mentioned in [7]. In the more recent past, some methods have been proposed to solve the power allocation problem. In particular, the authors in [8] investigated some game-theoretic approaches to solve the power allocation problem however without focusing on DPMRSs. The investigation of the power allocation only for localization is treated in [9], [10] and [11]. The capacity bound and the corresponding power allocation in a single relay system is considered in [12] and [13]. An optimal solution for the power allocation problem is given in [14], where an active radar is considered instead of a passive

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radar system. Moreover, an optimum power allocation scheme for decode-and-forward parallel relay networks, instead of amplify-and-forward sensor networks, is investigated in [15].

In the present work, we treat the power allocation problem in general and find analytical solutions in closed-form for DPMRSs. The first key idea is the utilization of the average deviation between the estimated and the actual signal values as a metric for defining the objective function. The second one is the application of a linear fusion rule to combine the distributed observations. We also use amplify-and-forward SNs in the network to obtain a simple system model. For optimizing the power allocation, total and individual power constraints are considered. Both lead to explicit policies for the power allocation. These are the main contributions of the present work.

We start with a description of the underlying technical system in the next section. Subsequently, the power allocation problem is specified and analytically solved. The achieved results are then discussed and carefully compared with each other.

Mathematical Notations:

Throughout this paper, we denote the sets of natural, integer, real and complex numbers by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. Note that the set of natural numbers does not include the element zero. Moreover, $\mathbb{R}_+$ denotes the set of non-negative real numbers. Furthermore, we use the subset $\mathbb{F}_N \subseteq \mathbb{N}$, which is defined as $\mathbb{F}_N := \{1, \ldots, N\}$ for any given natural number N. We denote the absolute value of a real or complex-valued number z by $|z|$. The expected value of a random variable v is denoted by $\mathcal{E}[v]$. Moreover, the notation $V^\star$ stands for the value of an optimization variable V at an optimum point of the corresponding optimization problem. Finally, vectors and matrices are represented in bold typeface.

II. Overview and Technical System Description

At any instance of time, a network of $K \in \mathbb{N}$ independent and spatially distributed sensors, as shown in Fig. 1, receives random observations. If a target signal r is present, then the received power at SN S_k is a part of the radiated power from the target source. Each received signal is weighted by the corresponding channel coefficient and is disturbed by additive noise. It is obvious that the sensing channel is wireless. All SNs continuously take samples from the disturbed received signal and amplify them without any additional data processing. The local measurements are then transmitted to a fusion center which is placed in a remote location. The communication to the fusion center is performed by using distinct waveforms for each SN so as to distinguish the communication of different SNs. Each waveform has to be suitably chosen in order to suppress inter-user (inter-node) interference at the fusion center. Hence, the K received signals at the fusion center are uncorrelated and are assumed to be conditionally independent. Each received signal at the fusion center is also weighted by the corresponding channel coefficient and is disturbed by additive noise, as well. The communication channel between the SNs and the fusion center can either be wireless or wired. The disturbed received signals

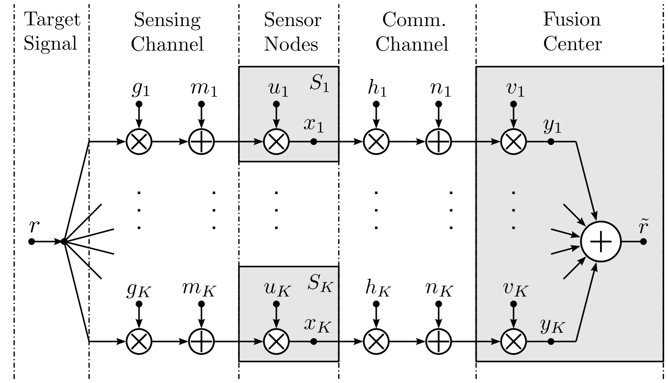

Fig. 2. System model of the distributed sensor network.

at the fusion center are weighted and combined together in order to obtain a single reliable observation $\tilde{r}$ of the actual target signal r.

Note that we disregard time delays within all transmissions and assume synchronized data communication.

In the following subsections, we mathematically describe the underlying system model that is depicted in Fig. 2. The continuous-time system is modeled by its discrete-time equivalent, where the sampling rate of the corresponding signals is equal to the target observation rate, for the sake of simplicity.

A. Target signal

Often, the target source is not well-known. Thus, we assume that we only know the quadratic mean $R := \mathcal{E}[|r|^2]$ with $0 < R < \infty$ of the complex-valued target signal r. This knowledge is sufficient for further calculations. Furthermore, the target signal during each observation step is assumed to be static.

B. Sensing channel

Each propagation path of the sensing channel is described by a corresponding random channel coefficient g_k. For the investigation of the power allocation problem, the concrete realization of the channel coefficients is needed and hence can also be used for postprocessing of the received signals at the SNs. We assume that the channel coefficients are complex-valued and static during each target observation step. Furthermore, the coherence time of communication channels is assumed to be much longer than the whole length of the classification process. Thus, the expected value and the quadratic mean of each coefficient during each observation step can be assumed to be equal to their instantaneous values, i.e., $\mathcal{E}[g_k] = g_k$ and $\mathcal{E}[|g_k|^2] = |g_k|^2$. In practice, it is often difficult to measure or estimate these coefficients because the network is passive and is hence not able to sound the channel actively. Thus, the results of the present work are applicable for scenarios where the channel coefficients can somehow be estimated accurately during each observation process or they are nearly deterministic and thus can be measured before starting the radar task. This is the case, for example for the neutrino telescope where the SNs are installed deep in the icecap.

Furthermore, the channel coefficients are assumed to be uncorrelated and jointly independent. Note that the channel coefficients include the radar cross section, the influence of the antenna, the impact of the filters, as well as all additional attenuation of the target signal.

At the input of each SN, the disturbance is modeled by the complex-valued additive white Gaussian noise (AWGN) m_k with zero mean and finite variance $M_k := \mathcal{E}[|m_k|^2]$ for all k. Note that the channel coefficient and the noise on the same propagation path are also uncorrelated and jointly independent.

C. Sensor nodes

We model each SN by an amplify-and-forward unit, where the ratio of the output to the input signal is described by the non-negative real-valued amplification factor u_k. Thus, the output signal and the expected value of its instantaneous power are described by

$$x_k := (rg_k + m_k)u_k\,,\ k \in \mathbb{F}_K \tag{1}$$

and

$$X_k := \mathcal{E}[|x_k|^2] = (R|g_k|^2 + M_k)u_k^2\,,\ k \in \mathbb{F}_K\,, \tag{2}$$

respectively. The average power consumption of each node is approximately equal to its average output power X_k, if the input signal is negligible in comparison to the output signal and if the nodes have smart power components with low-power dissipation loss. We assume that equality between X_k and the average power consumption of each node is ensured. In the present work, we also assume that the output power-range of each SN is individually limited by P_k and that the average power consumption of all SNs together is limited by the sum-power constraint P_{tot}. Hence, the constraints

$$X_k \le P_k \ \Leftrightarrow\ (R|g_k|^2 + M_k)u_k^2 \le P_k\,,\ k \in \mathbb{F}_K \tag{3}$$

and

$$\sum_{k=1}^{K} X_k \le P_{\text{tot}} \ \Leftrightarrow\ \sum_{k=1}^{K}(R|g_k|^2 + M_k)u_k^2 \le P_{\text{tot}} \tag{4}$$

arise consequently.

Note that the sum-power constraint P_{tot} is a reasonable approach to compare energy-efficient radar systems.

D. Communication channel

Analogous to the sensing channel, each propagation path of the communication channel is described by a corresponding random channel coefficient h_k. But in contrast to the sensing channel, we assume that the concrete realization of the communication channel coefficients is measurable by using pilot sequences at each SN. Accordingly, the channel coefficients can be used for postprocessing of received signals at the fusion center. We assume that the channel coefficients are complex-valued and static during each target observation step. Furthermore, the coherence time of communication channels is also assumed to be much longer than the whole length of the classification process. Thus, the expected value and the quadratic mean of each channel coefficient can be assumed to be equal to their instantaneous values, i.e., $\mathcal{E}[h_k] = h_k$ and $\mathcal{E}[|h_k|^2] = |h_k|^2$. Furthermore, the channel coefficients are assumed to be uncorrelated and jointly independent. Note that the channel coefficients include the influence of the antenna, the impact of the filters, as well as all additional attenuation of the corresponding sensor signal.

At the input of the fusion center, the disturbance on each communication path is modeled by the complex-valued AWGN n_k with zero mean and finite variance $N_k := \mathcal{E}[|n_k|^2]$ for all k. Note that the channel coefficient and the noise on the same propagation path are also uncorrelated and jointly independent.

E. Fusion center

The fusion center combines the different local observations into a single reliable one by applying a linear combiner. Thus, the received signals are weighted with the complex-valued factors v_k and summed up to yield an estimate $\tilde{r}$ of the actual target signal r. In this way, we obtain

$$y_k := \big((rg_k + m_k)u_k h_k + n_k\big)v_k\,,\ k \in \mathbb{F}_K\,, \tag{5}$$

and hence,

$$\tilde{r} := \sum_{k=1}^{K} y_k = r\sum_{k=1}^{K} g_k u_k h_k v_k + \sum_{k=1}^{K}(m_k u_k h_k + n_k)v_k\,. \tag{6}$$

Note that the fusion center can separate the input streams because the communication channel is either wired or the data communication is performed by distinct waveforms for each SN. Consequently, if the communication channel is wireless then a matched-filter bank is essential at the input of the fusion center to separate the data streams of different SNs.

In order to obtain a single reliable observation at the fusion center, the value $\tilde{r}$ should be a good estimate for the present target signal r. Thus, we optimize the amplification factors u_k and the weights v_k in order to minimize the average absolute deviation between $\tilde{r}$ and the true target signal r. This optimization procedure is elaborately explained in the next section.

F. Some remarks on the system model

All described assumptions are necessary to obtain a framework suitable for analyzing the power allocation problem, without studying detection, classification and estimation problems in specific systems and their settings.

The accurate estimation of all channel coefficients is necessary for both the radar process and the power allocation. Sometimes it is not possible to estimate the transmission channels; consequently the channel coefficients g_k and h_k remain unknown. In such cases, the radar usually fails to perform its task.

Since the channel coefficients g_k are in practice difficult to estimate or to determine, the present work rather shows theoretical aspects of the power allocation than the practical realization and implementation. Hence, the presented results act as theoretical bounds and references for comparing real radar systems.

Moreover, since the coherence time of communication channels as well as sensing channels is assumed to be much longer than the whole length of the classification process, the proposed power allocation method is applicable only for scenarios with slow fading channels.

Note that only the linear fusion rule together with the proposed objective function enable optimizing the power allocation in closed-form. The optimization of power allocation in other cases is in general hardly amenable analytically.

The introduced system architecture describes a baseband communication system without considering time, phase and frequency synchronization problems.

In order to distinguish the current operating mode of each SN in what follows, we say a SN is *inactive* or *idle* if the allocated power is zero. We say a SN is *active* if the allocated power is positive. Finally, we say a SN is *saturated* if the limitation of its output power-range is equal to the allocated power, i.e., $P_k = X_k$.

An overview of all notations that we will use hereinafter and are needed for the description of each observation process is depicted in Table I.

III. Power Optimization

In this section, we introduce the power optimization problem and consecutively present its analytical solutions for different power constraints. First, we investigate the case where the average transmission power of each SN is limited by the output power-range limitation $P_k \in \mathbb{R}_+$, $k \in \mathbb{F}_K$. Afterwards, we present the analytical solution of the power allocation problem for the case where only a sum-power constraint $P_{\text{tot}} \in \mathbb{R}_+$ for the cumulative sum of the expected power consumption of each SN is given. Finally, we extend the power allocation problem to the case where both constraints simultaneously hold and present the corresponding optimal solution.

A. The optimization problem

As mentioned in the last section, the value $\tilde{r}$ should be a good estimate for the present target signal r. In particular, we aim at finding estimators $\tilde{r}$ of minimum mean squared error in the class of unbiased estimators for each r.

The estimate $\tilde{r}$ is unbiased simultaneously for each r if $\mathcal{E}[\tilde{r} - r] = 0$, i.e., from equation (6) we obtain the identity

$$\sum_{k=1}^{K} g_k u_k h_k v_k = 1\,. \tag{7}$$

This identity is our first constraint in what follows. Note that the mean of the second sum in (6) vanishes since the noise is zero-mean. Furthermore, we do not consider the impact of both random variables g_k and h_k as well as their estimates in our calculations because the coherence time of both channels is assumed to be much longer than the target observation time. Note that equation (7) is complex-valued and may be separated as

$$\sum_{k=1}^{K} u_k |v_k g_k h_k| \cos(\vartheta_k + \phi_k) = 1 \tag{8}$$

and

$$\sum_{k=1}^{K} u_k |v_k g_k h_k| \sin(\vartheta_k + \phi_k) = 0\,, \tag{9}$$

where ϑ_k and ϕ_k are phases of v_k and $g_k h_k$, respectively.

The objective is to minimize the mean squared error $\mathcal{E}[|\tilde{r} - r|^2]$. By using equation (6) and the identity (7) we may write the objective function as

$$V := \mathcal{E}\big[|\tilde{r} - r|^2\big] = \sum_{k=1}^{K} \big(M_k u_k^2 |h_k|^2 + N_k\big)|v_k|^2\,. \tag{10}$$

Note that (10) is only valid if m_k and n_k are white and jointly independent.

As mentioned in the last section, each SN has an output power-range limitation and the expected overall power consumption is also limited. Hence, the objective function is also subject to (3) and (4), which are our second and last constraints, respectively.

In summary, the optimization problem is to minimize the mean squared error in (10) with respect to u_k and v_k, subject to constraints (3), (4), (8) and (9). Note that the optimization problem is a *signomial program*, which is a generalization of *geometric programming*, and is thus non-convex in general, see [16].

B. Power allocation subject to individual power constraint

In this subsection, we consider the power allocation problem only subject to the output power-range limitations from (3). In order to solve the optimization problem, we use the method of Lagrangian multipliers and obtain the corresponding constrained Lagrange function (relaxation with respect to the range of u_k and $|v_k|$) as

$$\begin{aligned} L_1(u_k, |v_k|, \vartheta_k; \eta_1, \eta_2, \lambda_k; \varrho_k) := &\sum_{k=1}^{K} \big(M_k u_k^2 |h_k|^2 + N_k\big)|v_k|^2 \\ &+ \left(1 - \sum_{k=1}^{K} u_k |v_k g_k h_k| \cos(\vartheta_k + \phi_k)\right)\eta_1 \\ &- \sum_{k=1}^{K} u_k |v_k g_k h_k| \sin(\vartheta_k + \phi_k)\eta_2 \\ &+ \sum_{k=1}^{K} \big(P_k - \varrho_k - (R|g_k|^2 + M_k)u_k^2\big)\lambda_k\,, \end{aligned} \tag{11}$$

where η_1, η_2 and λ_k are Lagrange multipliers while ϱ_k are slack variables.

Equation (9) is only then satisfied, if all phases $\vartheta_k + \phi_k$ are equal to $q_k \pi$, $q_k \in \mathbb{Z}$, for all $k \in \mathbb{F}_K$. If there were a better solution for $\vartheta_k + \phi_k$, then the first partial derivatives of L_1 with respect to ϑ_k would vanish for that solution, due to the continuity of trigonometric functions. But the first derivatives would lead to equations $\eta_1 \sin(\vartheta_k + \phi_k) = \eta_2 \cos(\vartheta_k + \phi_k)$ which cannot simultaneously satisfy both equations (8) and (9) for all η_1 and η_2. Thus, $q_k \pi$ is the unique solution. Hence, we may consequently write a modified Lagrange function as

$$\tilde{L}_1(u_k,|v_k|,q_k;\eta_1,\lambda_k;\varrho_k) \coloneqq \sum_{k=1}^{K}\big(M_k u_k^2|h_k|^2+N_k\big)|v_k|^2 + \left(1-\sum_{k=1}^{K}u_k|v_k g_k h_k|\cos(q_k\pi)\right)\eta_1 + \sum_{k=1}^{K}\big(P_k-\varrho_k-(R|g_k|^2+M_k)u_k^2\big)\lambda_k\,. \tag{12}$$

At any stationary point of $\tilde{L}_1$, all first partial derivatives must vanish, if they exist. This leads to

$$\frac{\partial\tilde{L}_1}{\partial|v_l|} = 2\big(M_l u_l^2|h_l|^2+N_l\big)|v_l| - \eta_1 u_l|g_l h_l|\cos(q_l\pi)=0\,,\ l\in\mathbb{F}_K\,, \tag{13}$$

$$\frac{\partial\tilde{L}_1}{\partial\eta_1} = 1-\sum_{k=1}^{K}u_k|v_k g_k h_k|\cos(q_k\pi)=0 \tag{14}$$

and

$$\frac{\partial\tilde{L}_1}{\partial\lambda_l} = P_l-\varrho_l-(R|g_l|^2+M_l)u_l^2=0\,,\ l\in\mathbb{F}_K\,. \tag{15}$$

Note that the first partial derivative with respect to u_l, $l\in\mathbb{F}_K$, is not needed because the optimal point lies on the boundary of the feasible set, as we will see later.

By multiplying (13) with $|v_l|$, summing up the outcome over all l, and using the identities (8) and (10), we obtain

$$\eta_1 = 2V \tag{16}$$

which is a positive real number due to definition of V. Because of the last relationship and according to (13), the value of $\cos(q_k\pi)$ must be a positive number and hence each q_k must be an even integer number. Thus, we can choose $q_k^\star=0$ for all $k\in\mathbb{F}_K$, without loss of generality, and conclude

$$\vartheta_k^\star = -\phi_k\,,\ k\in\mathbb{F}_K\,. \tag{17}$$

This solution gives the identity $\cos(q_k^\star\pi)=1$ which can be incorporated into (13) and (14).

From (13), we deduce the equation

$$|v_l| = \frac{\eta_1}{2}\frac{u_l|g_l h_l|}{M_l u_l^2|h_l|^2+N_l}\,. \tag{18}$$

Incorporating (18) into (14) yields the relationship

$$\frac{\eta_1}{2} = \left(\sum_{k=1}^{K}\frac{u_k^2|g_k h_k|^2}{M_k u_k^2|h_k|^2+N_k}\right)^{-1}. \tag{19}$$

In turn, we replace $\frac{\eta_1}{2}$ in (18) with (19) and obtain

$$|v_l| = \frac{u_l|g_l h_l|}{M_l u_l^2|h_l|^2+N_l}\left(\sum_{k=1}^{K}\frac{u_k^2|g_k h_k|^2}{M_k u_k^2|h_k|^2+N_k}\right)^{-1}. \tag{20}$$

Note that for each feasible u_k, $k\in\mathbb{F}_K$, equation (20) describes a feasible value for each $|v_k|$. Since for each $u_k>0$ the relation $|v_k|>0$ consequently follows, the feasible optimal values of each $|v_k|>0$ are not on the boundary $|v_k|=0$. Thus, finding optimal values for each u_k, $k\in\mathbb{F}_K$, leads to optimum values for each $|v_k|$, $k\in\mathbb{F}_K$, due to the convexity of (11) with respect to each $|v_k|$. Hence, finding a unique global optimum for u_k, $k\in\mathbb{F}_K$, yields the sufficient condition for the globally optimal solution of the minimization problem (11).

TABLE I
NOTATION OF SYMBOLS THAT ARE NEEDED FOR THE DESCRIPTION OF EACH OBSERVATION PROCESS.

Notation	Description
K	number of all nodes;
$\mathbb{F}_K$	the index-set of K nodes;
$\tilde{K}$	number of all active nodes;
$\mathbb{K}$	the index-set of all active nodes;
r, R	target signal and its quadratic absolute mean;
$\tilde{r}$	the estimate of r;
g_k, h_k	complex-valued channel coefficients;
m_k, n_k	complex-valued zero-mean AWGN;
M_k, N_k	variances of m_k and n_k;
u_k, v_k	non-negative amplification factors and complex-valued weights;
ϑ_k	phase of v_k;
ϕ_k	phase of $g_k h_k$;
y_k	input signals of the combiner;
X_k	output power of k^{th} SN;
P_k	output power-range limitation of k^{th} SN;
P_{tot}	sum-power constraint.

By considering (16) and (19), we deduce the identity

$$V = \frac{\eta_1}{2} = \left(\sum_{k=1}^{K}\frac{u_k^2|g_k h_k|^2}{M_k u_k^2|h_k|^2+N_k}\right)^{-1}, \tag{21}$$

where the objective and η_1 consequently are in terms of u_k. For the sake of simplicity and in order to compare the results later on, we define two new quantities as

$$\alpha_k \coloneqq \sqrt{\frac{|g_k|^2}{M_k}} \ \Rightarrow\ \alpha_k\in\mathbb{R}_+\,, \tag{22}$$

and

$$\beta_k \coloneqq \sqrt{\frac{N_k(R|g_k|^2+M_k)}{M_k|h_k|^2}} \ \Rightarrow\ \beta_k\in\mathbb{R}_+\,. \tag{23}$$

By using the new quantities as well as (2), the equation (21) is equivalent to

$$V = \frac{\eta_1}{2} = \left(\sum_{k=1}^{K}\frac{\alpha_k^2 X_k}{X_k+\beta_k^2}\right)^{-1}. \tag{24}$$

According to (2) and (15), we calculate the factors

$$u_l^2 = \frac{P_l-\varrho_l}{R|g_l|^2+M_l} \ \Leftrightarrow\ X_l = P_l-\varrho_l\,,\ l\in\mathbb{F}_K\,, \tag{25}$$

where ϱ_l is in the range $0\le\varrho_l\le P_l$. After replacing u_k^2 in (21), or X_k in (24), with (25), we obtain

$$V = \left(\sum_{l=1}^{K}\frac{\alpha_l^2}{1+\frac{\beta_l^2}{P_l-\varrho_l}}\right)^{-1}, \tag{26}$$

which is strictly increasing with respect to ϱ_l and strictly decreasing with respect to K. Thus, minimizing it leads to

$$\varrho_k^\star = 0 \ \Leftrightarrow\ X_k^\star = P_k\,,\ k\in\mathbb{F}_K\,, \tag{27}$$

$$u_k^\star = \sqrt{\frac{P_k}{R|g_k|^2+M_k}}\,,\ k\in\mathbb{F}_K\,, \tag{28}$$

and hence,

$$V^\star = \left(\sum_{k=1}^{K} \frac{\alpha_k^2 P_k}{P_k + \beta_k^2} \right)^{-1} . \tag{29}$$

By incorporating (28) into (20), we infer

$$|v_k^\star| = \frac{V^\star |g_k h_k| \sqrt{P_k} \sqrt{R|g_k|^2 + M_k}}{M_k |h_k|^2 P_k + N_k (R|g_k|^2 + M_k)} , \ k \in \mathbb{F}_K . \tag{30}$$

Since $\varrho_k^\star = 0$ for all $k \in \mathbb{F}_K$, it follows that the optimal point lies on the boundary of each u_k, $k \in \mathbb{F}_K$, especially on a corner, where the first derivatives of the objective with respect to u_k do not vanish in general.

The equations (17) and (27)–(30) describe the optimal solution of the power allocation problem only subject to the output power-range limitation per SN and hence are the main contribution of the present subsection.

Note that the global optimality of the obtained results is trivially reasoned, first because of the optimization of the relaxed Lagrange function (11) with extended range of all variables, and second since the global optimum point of the relaxed problem coincides with the real range of all variables.

C. Interpretation of the solution

The solution of the power allocation problem has the following interpretation: All K SNs are active and their output power is equal to their output power-range limitation P_k.

By using the amplification factors from (28) and the weights from (17) and (30), the single observation $\tilde{r}$ is an estimator of minimum mean squared error in the class of unbiased estimators for the target signal r. Hence, we obtain the estimate

$$\tilde{r} = r + \sum_{k=1}^{K} (m_k u_k^\star h_k + n_k) v_k^\star \tag{31}$$

from (6). The above equation shows that $\tilde{r}$ is equal to r with some additional noise. Hence, $\tilde{r} - r$ is a zero-mean Gaussian random variable with an absolute variance of $V^\star$, see (10) and (29).

Note that $\tilde{r}$ is an unbiased estimator for r due to constraint (7). By similar methods we can also minimize the mean squared error without restricting ourself to unbiased estimators. Obviously, the optimal value of V will then be smaller than that in (29).

D. Power allocation subject to sum-power constraint

In this subsection, we consider the power allocation problem only subject to the sum-power constraint from (4), which yields the constrained Lagrange function (relaxation with respect to the range of u_k and $|v_k|$)

$$\begin{aligned} L_2(u_k, |v_k|, \vartheta_k; \eta_1, \eta_2, \tau; \xi) := & \sum_{k=1}^{K} \big(M_k u_k^2 |h_k|^2 + N_k \big) |v_k|^2 \\ & + \left(1 - \sum_{k=1}^{K} u_k |v_k g_k h_k| \cos(\vartheta_k + \phi_k) \right) \eta_1 \\ & - \sum_{k=1}^{K} u_k |v_k g_k h_k| \sin(\vartheta_k + \phi_k) \eta_2 \\ & + \left(P_{\text{tot}} - \xi - \sum_{k=1}^{K} (R|g_k|^2 + M_k) u_k^2 \right) \tau \end{aligned} \tag{32}$$

with additional Lagrange multiplier τ and slack variable ξ.

The first partial derivatives of (32) with respect to $|v_l|$ and η_1 are identical to those which are given in (13) and (14), respectively. Thus, we also obtain the same results for ϑ_l, $|v_l|$ and V as given in (17), (20) and (24), respectively. Consequently, only the sum-power constraint remains unused, thus far.

Note that because of the same statement as mentioned in Subsection III-B, finding a unique global optimum for u_k, $k \in \mathbb{F}_K$, yields the sufficient condition for the globally optimal solution of the minimization problem (32).

Since the minimization of the objective V in (24) is equivalent to the minimization of $\tilde{V} := -V^{-1}$, we only consider the objective $\tilde{V}$ in the following. Initially, we highlight three important properties of $\tilde{V}$. First, the new objective function is strictly decreasing with respect to each X_k, $k \in \mathbb{F}_K$, which can easily be seen from the representation

$$\tilde{V} = - \sum_{k=1}^{K} \frac{\alpha_k^2}{1 + \beta_k^2 / X_k} . \tag{33}$$

Second, the objective function is twice differentiable with respect to each X_k, $k \in \mathbb{F}_K$, because its first and second derivatives exist. Third, the objective function is a jointly convex function with respect to $(X_k)_{k \in \mathbb{F}_K}$ which can be shown by calculating the corresponding Hessian $\mathbf{H} := (\frac{\partial^2 \tilde{V}}{\partial X_k \partial X_l})_{k,l \in \mathbb{F}_K}$. The Hessian is positive-definite because of

$$\begin{aligned} \mathbf{z}' \mathbf{H} \mathbf{z} = \sum_{k=1}^{K} \frac{2 \alpha_k^2 \beta_k^2 z_k^2}{(X_k + \beta_k^2)^3} > 0 , \\ \forall \ \mathbf{z} := (z_1, z_2, \ldots, z_K)' \in \mathbb{R}^K \setminus \{\mathbf{0}\} . \end{aligned} \tag{34}$$

By considering (2), we obtain that the remaining sum-power constraint in (32) is linear and thus also jointly convex with respect to $(X_k)_{k \in \mathbb{F}_K}$. Hence, we are able to define a modified convex minimization problem by the unconstrained Lagrangian

$$\begin{aligned} \tilde{L}_2(X_k; \varphi_k, \tau) := - \sum_{k=1}^{K} \frac{\alpha_k^2 X_k}{X_k + \beta_k^2} - \sum_{k=1}^{K} X_k \varphi_k \\ + \left(-P_{\text{tot}} + \sum_{k=1}^{K} X_k \right) \tau , \end{aligned} \tag{35}$$

where φ_k and τ are Lagrange multipliers. Note that the Lagrange multiplier η_1 is positive because of (16), and the equality $\sin(\vartheta_k^\star + \phi_k) = 0$ holds due to (17). Hence, both constraints (8) and (9) are discarded in (35). Furthermore, the sum-power constraint can be considered as an equality constraint instead of an inequality constraint due to monotonicity of the objective, see also complementary slackness theorem [17].

In order to solve the new convex optimization problem in (35), we apply the *Karush-Kuhn-Tucker* (KKT) conditions which are sufficient for optimality in convex problems, see [17]. These conditions are as follows for any optimal point $(X_k^\star, \varphi_k^\star, \tau^\star)$:

$$X_l^\star \geq 0\,,\ l \in \mathbb{F}_K\,, \tag{36a}$$

$$\varphi_l^\star \geq 0\,,\ l \in \mathbb{F}_K\,, \tag{36b}$$

$$X_l^\star \varphi_l^\star = 0\,,\ l \in \mathbb{F}_K\,, \tag{36c}$$

$$\sum_{k=1}^{K} X_k^\star = P_{\text{tot}}\,, \tag{36d}$$

and

$$\frac{\partial \tilde{L}_2}{\partial X_l} = -\frac{\alpha_l^2 \beta_l^2}{(X_l^\star + \beta_l^2)^2} - \varphi_l^\star + \tau^\star = 0\,,\ l \in \mathbb{F}_K\,. \tag{36e}$$

If $X_l^\star = 0$ for some $l \in \mathbb{F}_K$, then from (36b) and (36e) the inequality $\frac{1}{\sqrt{\tau^\star}} \leq \frac{\beta_l}{\alpha_l}$ follows. If $X_l^\star > 0$, then from (36c) and (36e) both the equality $X_l^\star = \alpha_l \beta_l \big(\frac{1}{\sqrt{\tau^\star}} - \frac{\beta_l}{\alpha_l}\big)$ and the inequality $\frac{1}{\sqrt{\tau^\star}} > \frac{\beta_l}{\alpha_l}$ follow. In summary, we may write

$$X_k^\star = \max\Big\{0, \alpha_k \beta_k \Big(\chi^\star - \frac{\beta_k}{\alpha_k}\Big)\Big\}\,,\ k \in \mathbb{F}_K\,, \tag{37}$$

where χ is a replacement for $\frac{1}{\sqrt{\tau}}$ and is called *water-level*. The water-level is implicitly determined by equation (36d) and gives the subset $\mathbb{K}$ of active SNs. By considering (37), we achieve the necessary and sufficient condition to select the right subset $\mathbb{K}$ of SNs for which the inequality $X_k^\star > 0$ with $k \in \mathbb{K}$ holds. Hence, all SNs for which the inequality $\chi^\star > \frac{\beta_k}{\alpha_k}$ holds are active. In order to determine their corresponding number $\tilde{K}$ as well as the water-level, we re-index all SNs such that the inequality chain

$$\frac{\beta_k}{\alpha_k} = \sqrt{\frac{N_k(R|g_k|^2 + M_k)}{|g_k h_k|^2}} \leq \frac{\beta_{k+1}}{\alpha_{k+1}}\,,\ k \in \mathbb{F}_{K-1}\,, \tag{38}$$

holds. Then, we can assume that the first $\tilde{K}$ SNs are members of $\mathbb{K} = \mathbb{F}_{\tilde{K}} \subseteq \mathbb{F}_K$. By inserting (37) into (36d), we obtain

$$\chi^\star = \frac{P_{\text{tot}} + \sum\limits_{k=1}^{\tilde{K}} \beta_k^2}{\sum\limits_{k=1}^{\tilde{K}} \alpha_k \beta_k}\,. \tag{39}$$

Due to the increasing order of the sequence $\frac{\beta_k}{\alpha_k}$ for all $k \in \mathbb{F}_K$, the inequality $\frac{\beta_{\tilde{K}}}{\alpha_{\tilde{K}}} < \chi^\star$ must hold for the last active SN. Thus, the number $\tilde{K}$ of active SNs is the largest integer number for which the inequality

$$P_{\text{tot}} > \sum_{k=1}^{\tilde{K}} \alpha_k \beta_k \Big(\frac{\beta_{\tilde{K}}}{\alpha_{\tilde{K}}} - \frac{\beta_k}{\alpha_k}\Big) \tag{40}$$

holds.

After incorporating $X_k^\star$ and $\chi^\star$ into (2), (24) and (20), we obtain

$$u_k^\star = \sqrt{\frac{1}{M_k |h_k|^2} \left(\chi^\star \sqrt{\frac{|g_k h_k|^2 N_k}{R|g_k|^2 + M_k}} - N_k\right)}\,,\ k \in \mathbb{K}\,, \tag{41}$$

$$V^\star = \left(\sum_{k=1}^{\tilde{K}} \alpha_k^2 - \frac{1}{\chi^\star} \sum_{k=1}^{\tilde{K}} \alpha_k \beta_k\right)^{-1}, \tag{42}$$

$$|v_k^\star| = \frac{V^\star u_k^\star |g_k h_k|}{M_k (u_k^\star)^2 |h_k|^2 + N_k}\,,\ k \in \mathbb{K}\,, \tag{43}$$

and

$$u_k^\star = v_k^\star = 0\,,\ k \in \mathbb{F}_K \setminus \mathbb{K}\,. \tag{44}$$

Note that by using the above results, the corresponding fusion rule is simplified by discarding the influence of inactive SNs from the fusion rule. The fusion rule (6) becomes

$$\tilde{r} = \sum_{k=1}^{\tilde{K}} y_k\,. \tag{45}$$

The equations (17), (37) and (41)–(44) together with (38)–(40) describe the optimal solution of the power allocation problem only subject to the sum-power constraint and hence are the main contribution of the present subsection.

As mentioned in Subsection III-B, the global optimality of the obtained results is also trivially reasoned, first because of the optimization of the relaxed Lagrange function (32) with extended range of all variables, and second since the global optimum point of the relaxed problem coincides with the real range of all variables.

E. Comparison of the solutions

In contrast to the case, where each SN has its individual output power-range limitation, only some of the SNs are active. In this case, the amount of the available sum-power is inadequate to supply all SNs at their output power-range limitation. Hence, the available sum-power can only be allocated to those SNs, which are members of the subset $\mathbb{K}$, while all other SNs remain inactive, since their information reliability is too poor to be considered for data fusion. The best SNs are those which have the smallest ratio of $\frac{\beta_k}{\alpha_k}$ that can be interpreted as *interference power*. This means that for the identification of the most reliable SNs in a certain network, that can be modeled as depicted in Fig. 2, only the ratios $\frac{\beta_k}{\alpha_k}$ are important. As one can see from (38), the best SNs have the largest absolute values of channel coefficients as well as the smallest noise powers. Consequently, SNs which are placed on the straight line between the target source and the fusion center are more reliable than SNs which are not placed on that line and are far from the target source as well as from the fusion center.

Note that the obtained results are quite similar but not identical to the well-known *water-filling* solution, see [18]. The distinction arises from our definition of the water-level χ which differs from the general description.

F. Optimization subject to all constraints

In the current subsection, we consider the optimization problem from Subsection III-A subject to all constraints, i.e., sum-power constraint as well as output power-range limitation per SN. Two of three different cases can be singled out and reduced to preceding instances.

First, if $\sum_{k\in\mathbb{F}_K} P_k \leq P_{\text{tot}}$, then the sum-power constraint is irrelevant, because the feasible set is only limited by the individual output power-range constraints. Hence, the power allocation problem reduces to the one described in Subsection III-B with results given in (17) and (27)–(30). The only difference is that possibly a part of the available sum-power remains unallocated and cannot be used.

Secondly, if $P_{\text{tot}} \leq \min\limits_{k\in\mathbb{F}_K}\{P_k\}$, then the individual output power-range constraints are irrelevant, because the feasible set is only limited by the sum-power constraint. Hence, the power allocation problem is equal to the one described in Subsection III-D with results given in (17), (37) and (41)–(44) with (38)–(40).

The case of $\min\limits_{k\in\mathbb{F}_K}\{P_k\} < P_{\text{tot}} < \sum_{k\in\mathbb{F}_K} P_k$ is the most challenging one. The amount of available sum-power is on the one hand inadequate to supply all SNs at their output power-range limitation. Hence, the available sum-power can only be allocated to some of the SNs while all others remain inactive, as we will see later. On the other hand, some of the SNs can attain the limit of their individual output power-range and are thus saturated. Therefore, we have to separate all active SNs into two groups. The first group contains all active SNs, which are saturated, and is denoted by the subset $\mathbb{K}_{\text{sat}}$. The second group contains all other active SNs, which operate within their output power-range, and is denoted by the subset $\mathbb{K}_{\text{lin}}$. Note that both subsets are disjoint and their union is the subset of all active SNs, i.e., $\mathbb{K} = \mathbb{K}_{\text{sat}} \cup \mathbb{K}_{\text{lin}}$ and $\mathbb{K}_{\text{sat}} \cap \mathbb{K}_{\text{lin}} = \emptyset$ with $\mathbb{K} \subseteq \mathbb{F}_K$.

Since the optimization problem under investigation is the same as in Subsection III-D with additional constraints, $X_k \leq P_k$ for all $k \in \mathbb{K}$, the first few problem-solving steps are equal to those described in Subsection III-D. Thus, we can start to formulate an extended convex minimization problem by the unconstrained Lagrangian

$$\tilde{L}_3(X_k;\varphi_k,\lambda_k,\tau) := -\sum_{k\in\mathbb{F}_K}\frac{\alpha_k^2 X_k}{X_k+\beta_k^2} - \sum_{k\in\mathbb{F}_K} X_k\varphi_k + \sum_{k\in\mathbb{F}_K}(-P_k+X_k)\lambda_k + \Bigg(-P_{\text{tot}} + \sum_{k\in\mathbb{F}_K} X_k\Bigg)\tau\,. \tag{46}$$

In order to solve the problem in (46), we again apply the KKT conditions which are as follows, for any optimal point $(X_k^\star,\varphi_k^\star,\lambda_k^\star,\tau^\star)$:

$$X_l^\star \geq 0\,,\ l\in\mathbb{F}_K\,, \tag{47a}$$
$$X_l^\star \leq P_l\,,\ l\in\mathbb{F}_K\,, \tag{47b}$$
$$\varphi_l^\star \geq 0\,,\ l\in\mathbb{F}_K\,, \tag{47c}$$
$$\lambda_l^\star \geq 0\,,\ l\in\mathbb{F}_K\,, \tag{47d}$$
$$X_l^\star\varphi_l^\star = 0\,,\ l\in\mathbb{F}_K\,, \tag{47e}$$
$$(X_l^\star - P_l)\lambda_l^\star = 0\,,\ l\in\mathbb{F}_K\,, \tag{47f}$$

Algorithm 1 Separation of active sensor nodes

$\mathbb{K}_{\text{sat}} \leftarrow \emptyset$
$P_{\text{remain}} \leftarrow P_{\text{tot}}$
repeat
  $\mathbb{K}_{\text{lin}} \leftarrow \mathbb{F}_K \setminus \mathbb{K}_{\text{sat}}$
  repeat
    $\chi_{\mathbb{K}} \leftarrow \frac{P_{\text{remain}}+\sum_{k\in\mathbb{K}_{\text{lin}}}\beta_k^2}{\sum_{k\in\mathbb{K}_{\text{lin}}}\alpha_k\beta_k}$ ▷ see (49)
    $X_k \leftarrow \alpha_k\beta_k\big(\chi_{\mathbb{K}} - \frac{\beta_k}{\alpha_k}\big)\,,\ k\in\mathbb{K}_{\text{lin}}$ ▷ see (48)
    $\mathbb{K}_- \leftarrow \{k\in\mathbb{K}_{\text{lin}} \mid X_k \leq 0\}$
    $\mathbb{K}_{\text{lin}} \leftarrow \mathbb{K}_{\text{lin}} \setminus \mathbb{K}_-$
  until $\mathbb{K}_- = \emptyset$ **or** $\mathbb{K}_{\text{lin}} = \emptyset$
  $\mathbb{K}_+ \leftarrow \{k\in\mathbb{K}_{\text{lin}} \mid X_k \geq P_k\}$
  $\mathbb{K}_{\text{sat}} \leftarrow \mathbb{K}_{\text{sat}} \cup \mathbb{K}_+$
  $P_{\text{remain}} \leftarrow P_{\text{remain}} - \sum_{k\in\mathbb{K}_+} P_k$
until $\mathbb{K}_+ = \emptyset$ **or** $\mathbb{K}_{\text{sat}} = \mathbb{F}_K$
$\mathbb{K}_{\text{lin}} \leftarrow \mathbb{K}_{\text{lin}} \setminus \mathbb{K}_+$
return $(\mathbb{K}_{\text{lin}},\mathbb{K}_{\text{sat}})$

$$\sum_{k\in\mathbb{F}_K} X_k^\star = P_{\text{tot}}\,, \tag{47g}$$

and

$$\frac{\partial\tilde{L}_3}{\partial X_l} = -\frac{\alpha_l^2\beta_l^2}{(X_l^\star+\beta_l^2)^2} - \varphi_l^\star + \lambda_l^\star + \tau^\star = 0\,,\ l\in\mathbb{F}_K\,. \tag{47h}$$

If $X_l^\star = 0$ for some $l\in\mathbb{F}_K$, then from (47c), (47f) and (47h) the inequality $\frac{1}{\sqrt{\tau^\star}} \leq \frac{\beta_l}{\alpha_l}$ follows. If $X_l^\star = P_l$ for some $l\in\mathbb{F}_K$, then from (47d), (47e) and (47h) the inequality $\frac{1}{\sqrt{\tau^\star}} \geq \frac{P_l+\beta_l^2}{\alpha_l\beta_l}$ follows. If $0 < X_l^\star < P_l$, then from (47e), (47f) and (47h) both the equality $X_l^\star = \alpha_l\beta_l\big(\frac{1}{\sqrt{\tau^\star}} - \frac{\beta_l}{\alpha_l}\big)$ and the inequality $\frac{1}{\sqrt{\tau^\star}} > \frac{\beta_l}{\alpha_l}$ follow. In summary, we may write

$$X_k^\star = \max\Big\{0, \min\Big\{P_k, \alpha_k\beta_k\Big(\chi_{\mathbb{K}}^\star - \frac{\beta_k}{\alpha_k}\Big)\Big\}\Big\},\ k\in\mathbb{F}_K, \tag{48}$$

where $\chi_{\mathbb{K}}$ is again a replacement for $\frac{1}{\sqrt{\tau}}$. The water-level $\chi_{\mathbb{K}}$ depends on the subset $\mathbb{K}$ of active SNs and can iteratively be determined by equation (47g), as we will show later on. By considering (48), we achieve the necessary and sufficient condition to select the right subset $\mathbb{K}$ of SNs for which the inequality $X_k^\star > 0$ with $k\in\mathbb{K}$ holds. Hence, all SNs for which the inequality $\chi_{\mathbb{K}}^\star > \frac{\beta_k}{\alpha_k}$ holds are active. In order to determine the corresponding water-level, we insert (48) into (47g) and infer

$$\chi_{\mathbb{K}}^\star = \frac{P_{\text{tot}} - \sum\limits_{k\in\mathbb{K}_{\text{sat}}} P_k + \sum\limits_{k\in\mathbb{K}_{\text{lin}}}\beta_k^2}{\sum\limits_{k\in\mathbb{K}_{\text{lin}}}\alpha_k\beta_k}\,. \tag{49}$$

As one can see, for calculation of the water-level both subsets $\mathbb{K}_{\text{sat}}$ and $\mathbb{K}_{\text{lin}}$ are needed, and vice versa, we need the water-level to determine the subset of all active SNs. Thus, if we are able to determine which SNs are active, and in turn, which of them are saturated, then we can continue solving the optimization problem in (46). It is possible to sort the SNs by $\frac{\beta_k}{\alpha_k}$ in ascending order again and extend the approach from Subsection III-D by particular consideration on saturated SNs, as in (48). However, we want to present an efficient algorithm which avoids the sorting of SNs. Note that the proposed algorithm can be implemented more efficiently, but for sake of comprehensibility, we have chosen the given representation. In the following, we will describe and show that Algorithm 1 optimally determines both subsets $\mathbb{K}_{\text{sat}}$ and $\mathbb{K}_{\text{lin}}$ of active SNs.

First, the results from Subsection III-D are applied in the inner loop to achieve an optimal solution neglecting the individual output power-range constraints. In the first repetition, this is performed on all SNs and in each further repetition on all SNs included in the subset $\mathbb{K}_{\text{lin}}$ in order to determine all active SNs of the current repetition. At the end of the inner loop, $\mathbb{K}_{+}$ contains all SNs which operate at their individual output power-range limitation. They are added to the subset of all saturated SNs $\mathbb{K}_{\text{sat}}$. At last, the power used by those SNs is subtracted from the available sum-power which gives the remaining sum-power P_{remain}. With these updated settings, the procedure is repeated until, for a new set of active SNs, the subset $\mathbb{K}_{+}$ of saturated SNs is empty. Note that $\mathbb{K}_{\text{sat}}$ might be empty. We will show later on, that the water-level, and thereby, the power for each non-saturated SN is increasing in each repetition of the outer loop. Thus, it is possible that SNs may become active, and hence, all non-saturated SNs are potential active candidates. Finally, we get the (optimal) subsets of active SNs to continue solving the optimization problem in (46).

After determination of $\mathbb{K}_{\text{sat}}$ and $\mathbb{K}_{\text{lin}}$, we use (49) to calculate $\chi^{\star}_{\mathbb{K}}$, and subsequently, by inserting $\chi^{\star}_{\mathbb{K}}$ into (48) we obtain $X^{\star}_{k}$. In turn, from (2), (24) and (20), we infer

$$u^{\star}_{k} = \sqrt{\frac{P_k}{R|g_k|^2 + M_k}}\,,\ k \in \mathbb{K}_{\text{sat}}\,, \tag{50}$$

$$u^{\star}_{k} = \sqrt{\frac{1}{M_k|h_k|^2}\left(\chi^{\star}_{\mathbb{K}}\sqrt{\frac{|g_k h_k|^2 N_k}{R|g_k|^2 + M_k}} - N_k\right)}\,,\ k \in \mathbb{K}_{\text{lin}}\,, \tag{51}$$

$$V^{\star} = \left(\sum_{k\in\mathbb{K}_{\text{sat}}} \frac{\alpha_k^2 P_k}{P_k + \beta_k^2} + \sum_{k\in\mathbb{K}_{\text{lin}}} \alpha_k^2 - \frac{1}{\chi^{\star}_{\mathbb{K}}} \sum_{k\in\mathbb{K}_{\text{lin}}} \alpha_k\beta_k\right)^{-1}, \tag{52}$$

$$|v^{\star}_{k}| = \frac{V^{\star} u^{\star}_{k} |g_k h_k|}{M_k (u^{\star}_{k})^2 |h_k|^2 + N_k}\,,\ k \in \mathbb{K}\,, \tag{53}$$

and

$$u^{\star}_{k} = v^{\star}_{k} = 0\,,\ k \in \mathbb{F}_K \setminus \mathbb{K}\,. \tag{54}$$

The equations (17), (48) and (50)–(54) together with (49) and Algorithm 1 describe the optimal solution of the power allocation problem and hence are the main contribution of the present subsection. Note that the obtained results are obviously mixtures of both solutions from Subsection III-B and III-D.

An example for the described power allocation is depicted in Fig. 3. Obviously, Algorithm 1 terminates and gives a feasible solution. Moreover, the final water-level is as in (49) if $\mathbb{K}_{\text{sat}}$ is optimally determined. However, $\mathbb{K}_{\text{sat}}$ is easily given for all SNs, because the condition $\chi^{\star}_{\mathbb{K}} \geq \frac{P_l + \beta_l^2}{\alpha_l \beta_l}$ is satisfied for all saturated SNs if the water-level is increasing in each step of the outer loop. Hence, for optimality of Algorithm 1, it only remains to show that in each repetition the water-level is increasing in the outer loop. Note that in contrast the water-level is decreasing in the inner loop in each repetition. These statements are discussed in the following.

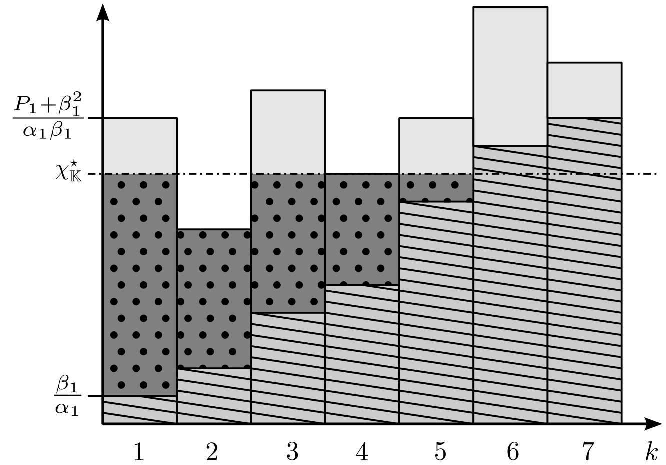

Fig. 3. An example of the power allocation for $K = 7$ sensor nodes is shown. The dotted dark area is the allocated power. The sensor nodes are ascendingly ordered with respect to interference powers β_k/α_k. The striped area is the interference power. The bright shaded area is the remaining part of the available output power-range for each sensor node. The water-level $\chi^{\star}_{\mathbb{K}}$ is indicated by the dashed line, where the number of active sensor nodes is equal to $\tilde{K} = 5$. Sensor 2 and 4 are saturated, while 6 and 7 are inactive.

Whenever $X_l \geq P_l$ holds for a specific $l \in \mathbb{K}_{\text{lin}}$, it follows from (48) the inequality

$$X_l = \alpha_l\beta_l\left(\chi_{\mathbb{K}} - \frac{\beta_l}{\alpha_l}\right) \geq P_l \ \Rightarrow\ -P_l - \beta_l^2 \geq -\alpha_l\beta_l\chi_{\mathbb{K}}\,. \tag{55}$$

By using $\tilde{\mathbb{K}} := \tilde{\mathbb{K}}_{\text{lin}} \cup \tilde{\mathbb{K}}_{\text{sat}}$ with $\tilde{\mathbb{K}}_{\text{lin}} := \mathbb{K}_{\text{lin}} \setminus \{l\}$ and $\tilde{\mathbb{K}}_{\text{sat}} := \mathbb{K}_{\text{sat}} \cup \{l\}$, the definition (49), and the inequality (55), it easily leads to

$$\begin{aligned}
\chi_{\tilde{\mathbb{K}}} &= \frac{P_{\text{tot}} - \sum\limits_{k\in\tilde{\mathbb{K}}_{\text{sat}}} P_k + \sum\limits_{k\in\tilde{\mathbb{K}}_{\text{lin}}} \beta_k^2}{\sum\limits_{k\in\tilde{\mathbb{K}}_{\text{lin}}} \alpha_k\beta_k} \\
&= \frac{-P_l - \beta_l^2 + P_{\text{tot}} - \sum\limits_{k\in\mathbb{K}_{\text{sat}}} P_k + \sum\limits_{k\in\mathbb{K}_{\text{lin}}} \beta_k^2}{-\alpha_l\beta_l + \sum\limits_{k\in\mathbb{K}_{\text{lin}}} \alpha_k\beta_k} \\
&\geq \frac{P_{\text{tot}} - \sum\limits_{k\in\mathbb{K}_{\text{sat}}} P_k + \sum\limits_{k\in\mathbb{K}_{\text{lin}}} \beta_k^2}{\sum\limits_{k\in\mathbb{K}_{\text{lin}}} \alpha_k\beta_k} = \chi_{\mathbb{K}}\,.
\end{aligned} \tag{56}$$

This means that omitting the power exceeding SN l and recalculating the water-level leads to an increased water-level and in turn to more transmission power for all remaining SNs. It might even turn out that new SNs may be included in the subset $\mathbb{K}_{\text{lin}}$ of active candidates. However, this will just slow down, but not stop the increase of the water-level. In summary, the individual transmission power of each SN, which is a member of $\mathbb{K}_{\text{lin}}$, will be higher than in the previous loop. The same argumentation can now be used to show that omitting the SN l with non-positive allocated power and recalculating the water-level leads to a decreased water-level.

Whenever $X_l \leq 0$ holds for a specific $l \in \mathbb{K}_{\text{lin}}$, it follows from (48) the inequality

$$X_l = \alpha_l\beta_l\left(\chi_{\mathbb{K}} - \frac{\beta_l}{\alpha_l}\right) \leq 0 \ \Rightarrow\ -\beta_l^2 \leq -\alpha_l\beta_l\chi_{\mathbb{K}}\,. \tag{57}$$

By using the above set definitions, the definition (49), and the inequality (57), it easily leads to

$$\chi_{\mathbb{K}\setminus\{l\}} = \frac{P_{\text{tot}} - \sum\limits_{k\in\mathbb{K}_{\text{sat}}} P_k + \sum\limits_{k\in\tilde{\mathbb{K}}_{\text{lin}}} \beta_k^2}{\sum\limits_{k\in\tilde{\mathbb{K}}_{\text{lin}}} \alpha_k\beta_k}$$
$$= \frac{-\beta_l^2 + P_{\text{tot}} - \sum\limits_{k\in\mathbb{K}_{\text{sat}}} P_k + \sum\limits_{k\in\mathbb{K}_{\text{lin}}} \beta_k^2}{-\alpha_l\beta_l + \sum\limits_{k\in\mathbb{K}_{\text{lin}}} \alpha_k\beta_k}$$
$$\leq \frac{P_{\text{tot}} - \sum\limits_{k\in\mathbb{K}_{\text{sat}}} P_k + \sum\limits_{k\in\mathbb{K}_{\text{lin}}} \beta_k^2}{\sum\limits_{k\in\mathbb{K}_{\text{lin}}} \alpha_k\beta_k} = \chi_{\mathbb{K}} \,. \quad (58)$$

Note that increasing the power of inactive SNs cannot lead to an improvement of our solution, because this would contradict the results from Subsection III-D.

G. Discussion of the solutions

The main difficulty and difference between both solutions from Subsection III-D and III-F arises from the individual output power-range limitation per SN. In both cases, the operating mode of each SN mainly depends on its corresponding interference-level $\frac{\beta_k}{\alpha_k}$ which is easily visible from (37) and (48). In Subsection III-F, we have given an efficient algorithm which avoids the sorting of SNs by their interference-level. Moreover, the same algorithm might be applied to the problem in Subsection III-D, as well. In practice, it is hence more complicated to calculate the optimal solution of the power allocation problem from Subsection III-F than that from Subsection III-D. Eventually, the complexity of obtained results is not surprising, because all discussed optimization problems are signomial programs, as mentioned in Subsection III-A. Nevertheless, they lead to convex optimization problems which are analytically solvable in closed-form.

IV. Conclusion

The main contribution of the present work is an optimal solution to the power allocation problem for increasing the system performance of distributed passive multiple-radar systems while the power consumption of the whole sensor network is kept constant. We have introduced a system model, a linear fusion rule and a simple objective function, which enable us to solve the power allocation problem analytically. Three different cases of power constraints have been investigated. For a limitation of transmission power per sensor node as well as for a sum-power limitation, we have analytically obtained optimal solutions in closed-form. We have seen that the power allocation problem is harder to solve if both constraints shall simultaneously be satisfied. Hence, we have developed an efficient algorithm to solve the last problem optimally. The proposed results enable us to calculate the optimal power allocation fast and accurately which is essential for distributed passive multiple-radar systems with a large number of sensor nodes.

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