Power Control, Capacity, and Duality of Uplink and Downlink in Cellular CDMA Systems

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***Abstract*—Accurate power control is an essential requirement in the design of cellular code-division multiple-access (CDMA) systems. In this paper, we contribute three main themes to the power control problem. First, we derive an efficient algorithm for computing minimal power levels for large-scale networks within seconds. Nice and intuitive conditions for the existence of feasible power solutions follow from this approach. Second, we define the capacity region of a network by the set of effective spreading gains, or data rates, respectively, which can be supplied by the network. This is achieved by bounding the spectral radius of a certain matrix containing system parameters and mutual transmission gain information. It is shown that the capacity region is a convex set. Finally, we reveal an interesting duality between the uplink and downlink capacity region. In a clear-cut analytical way, it substantiates the fact that the uplink is the more restricting factor in cellular radio networks. The same methods carry over to certain models of soft handover. In the case that the channel gains are subject to log-normal shadowing, we introduce the concept of level-α capacity regions. Despite the complicated structure, it can still be shown that this set is sandwiched by two convex sets coming arbitrarily close as variance decreases.**

***Index Terms*—Capacity region, cellular networks, code-division multiple access (CDMA), convexity, log-normal fading, soft handover.**

I. Introduction

The upcoming Universal Mobile Telecommunication System (UMTS) in Europe has initiated a lot of research going into the capacity of code-division multiple-access (CDMA) technology with a constant chip rate when data rates as well as link quality may vary individually. Optimal uplink and downlink power control is the vehicle for achieving diverse quality of service (QoS) and rate requirements and for keeping interference, electromagnetic radiation, and power consumption low. One objective of this paper is to represent the capacity of an n-user CDMA system by characterizing the set of data rates and QoS demands which are achievable under optimal power control. Another purpose is to provide a fast and efficient algorithm for computing the optimal power allocation, even for large-scale mobile networks.

Early analytical work on power control is reported in [17], where power control is considered as a min–max interference balancing problem, however, in the absence of noise. In a comment on this paper, it is shown by the same author [18] that the maximum achievable signal-to-interference ratio (SIR) for uplink and downlink are identical if the channel gain is symmetric. Optimization theory is also employed in the recent paper [14] for tackling the power control problem. The objective here is to maximize the total effective rate in a system with users of different data rates and quality demands.

One of the first papers to address the power control and assignment problem for CDMA by considering systems of linear equations with positive solutions is [3]. In this paper, the existence of a feasible power-control vector is clarified by use of Perron–Frobenius' theory. Furthermore, in [3], a provably convergent algorithm is presented for assigning mobiles to base stations. In [16], the power control and base station assignment problem is addressed as an integrated optimization setup.

In an excellent survey paper, the authors of [6] consider power control as a flexible mechanism to ensure QoS demands of individual users. Mainly, two questions are studied, optimal power control and characterizing the resulting network capacity under different receiver designs.

The authors of [11] develop this idea further by formulating the QoS aspect of power control as a noncooperative game, where users want to maximize their utility function. The concept of Nash equilibrium is introduced, and transmit power is priced to achieve a Pareto improvement.

The recent work of [2] employs a discrete-time Markov chain to model time-variant channel gain. Dynamic multiuser power control is formulated in the framework of Markov decision processes with an objective function trading off between a minimal total received power and certain functionals of transmission quality.

Section II of this paper deals with the dimensionality reduction of the system of linear equations for finding a solution to the power control problem. Related approaches, however, with different agglomerated variables are given in [5], and subsequently, [9]. In our approach, cellwise lumped variables are used, which besides dimensionality reduction yields a nice intuitive condition for the nonexistence of a feasible power assignment.

Parts of our methodology are related to the work in [1]. The carrier-to-interference ratio (CIR) is of importance also in the beamforming concept. The crosstalk terms between stations depend on the channel and beamforming vectors and relate to the transmission gain coefficients in this paper.

There are three main themes in our contribution to the power control problem. First, we develop a fast offline power-allocation algorithm that proceeds in two steps. The between-cells interference relations are solved by means of a relatively

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TABLE I
BASIC NOTATIONS

$k \in \{1,\ldots,K\}$	Labeling of base stations
$i \in \{1,\ldots,n\}$	Labeling of mobile stations
k_i	Base station allocated to mobile i
$\mathcal{C}(k) \subseteq \{1,\ldots,n\}$	Set of mobiles allocated to base station k
ω	Chip rate of the CDMA system
R_i	Data rate of user i
$s_i = \omega/R_i$	Spreading gain of user i
e_i	Minimum acceptable bit error rate of user i
$s_i' = s_i/e_i$	Effective spreading gain of user i
p_i	Uplink transmit power of user i
q_i	Downlink transmit power for user i
τ_k^0	General background noise and pilot signal power at base station k
σ_i^0	General background noise and pilot signal power at mobile station i
$a_{ik} \in [0,1]$	Transmission gain from mobile i to base station k

small system of linear equations, with dimension equal to the number of base stations. Once a solution is obtained, the individual power level of mobiles can be computed explicitly. This approach also allows for characterizing the existence of an admissible power allocation in a nice, intuitive manner. The algorithm is particularly suited for integration into cell site-selection algorithms and planning tools for cellular radio networks. Second, we define the capacity region by a set of user demand vectors, in terms of data rate and QoS requirements which admit a feasible power control scheme. It is shown that this set is convex for uplink and downlink. Finally, we work out an interesting duality between the uplink and downlink capacity regions. This substantiates the fact that the uplink is the more restricting factor from a capacity point of view.

It is shown that the very same methods carry over to certain models of soft and softer handover. To also include log-normal shadowing into our investigations, we introduce the concept of the level-α capacity region, which contains the set of feasible user requirements that can be served by the network with probability $1-\alpha$. A sandwich theorem with convex layering sets shows that the random capacity region is close to convexity.

Technical proofs of the paper are collected in the Appendix.

II. EXISTENCE AND EFFICIENT COMPUTATION OF POWER CONTROL

We start by introducing the basic notation (for an overview, see Table I). Assume a CDMA system with chip rate ω, e.g., $\omega = 3.84$ MChip/s for UMTS. Each user $i \in \{1,\ldots,n\}$ has a certain data rate R_i to transmit and requires an individual minimum bit-error rate (BER). Let $s_i = \omega/R_i$ denote the spreading gain. Since the BER rate is a function of the bit energy-to-noise ratio, E_b/N_0, individual quality demands can be described by lower bounds e_i as follows:

$$\left(\frac{E_b}{N_0}\right)_i = s_i \left(\frac{C}{I}\right)_i \geq e_i \tag{1}$$

where C/I denotes the CIR at the mobile's connecting base station.

In the following, we assume a fixed allocation of mobiles to base stations, expressed by an assignment function

$$c : \{1,\ldots,n\} \to \{1,\ldots,K\} : i \mapsto k_i$$

such that k_i denotes i's connecting base station. The set of mobiles allocated to base station k is denoted by $\mathcal{C}(k) = \{i | k_i = k\}$, $k = 1,\ldots,K$. $\mathcal{C}(k)$ is simply a partition of the set $\{1,\ldots,n\}$.

In the uplink, let p_i denote the transmit power of mobile i and $a_{ik} \in [0,1]$ the transmission gain from mobile i to base station k. Activity periods are also included in this model by understanding a_{ik} as a product of the transmission gain itself and a potential activity factor of station i. We assume that $a_{ik} > 0$ for all $i \in \mathcal{C}(k)$, which is obvious to avoid meaningless assignments. Using the *effective spreading gain* $s_i' = s_i/e_i$, (1) reads as

$$s_i' \left(\frac{C}{I}\right)_i = s_i' \frac{p_i a_{ik_i}}{\sum_{j \neq i} p_j a_{jk_i} + \tau_{k_i}^0} \geq 1, \quad i = 1,\ldots,n. \tag{2}$$

The numerator $p_i a_{ik_i}$ represents the received power of mobile i at the connecting base station k_i, $\sum_{j \neq i} p_j a_{jk_i}$ collects the received interference from all other mobiles, and $\tau_{k_i}^0 > 0$ denotes the general background and thermal receiver noise at base station k_i. It also includes the system's pilot signal pollution.

The problem now is to determine the minimum transmit power for mobiles such that (2) is satisfied. Since the numerator of (2) is increasing in p_i and the denominator is increasing in p_j, $j \neq i$, it is clear that the minimum is attained at the boundary such that a solution $\boldsymbol{p} = (p_i)_{1 \leq i \leq n}$ of the system

$$s_i' \frac{p_i a_{ik_i}}{\sum_{j \neq i} p_j a_{jk_i} + \tau_{k_i}^0} = 1, \quad i = 1,\ldots,n \tag{3}$$

is needed. Equation (3) is easily converted into the following system of linear equations:

$$s_i' a_{ik_i} p_i - \sum_{j \neq i} a_{jk_i} p_j = \tau_{k_i}^0, \quad i = 1,\ldots,n. \tag{4}$$

The system of (3) and (4), respectively, is the common starting point for work on CDMA power control. However, the number of mobiles is usually large, such that several thousand equations may be involved. Dimensionality reduction is an important issue, also addressed in two related papers [5], [9]. The author of [5] calls system (4) the microscopic view of the problem, accepts a slight inaccuracy by allowing "self-interference," and reduces dimension by switching to the lumped variables $Q[k] = \sum_{u \in \mathcal{C}(k)} p_u a_{uk}$ (in this paper's notation). Using an easy transformation, the authors of [9] are able to eliminate the self-interference approximation in [5].

The dimension of system (4) is then reduced by employing the agglomerated variables $R(k) = \sum_{i=1}^{n} a_{ik}p_k + \tau_k^0$.

We choose a different set of variables for dimensionality reduction. Interestingly, our approach also leads to a nice and intuitive characterization of the nonexistence of any feasible power allocation.

For this purpose, we select some base station $k \in \{1,\dots,K\}$ and rewrite (4) as

$$s_i' a_{ik} p_i - \sum_{j\in\mathcal{C}(k)\setminus\{i\}} a_{jk}p_j = \tau_k, \quad i \in \mathcal{C}(k) \tag{5}$$

where $\tau_k = \tau_k^0 + \sum_{j\notin\mathcal{C}(k)} a_{jk}p_j$ is the interference at base station k, composed of the background noise and the interference from mobiles in other cells.

Proposition 1: The unique solution of system (5) is given by

$$p_i = \frac{\tau_k}{a_{ik}\left(s_i'+1\right)\left(1-\sum_{j\in\mathcal{C}(k)}\frac{1}{s_j'+1}\right)} = \gamma_i(k)\tau_k, \;\; i\in\mathcal{C}(k). \tag{6}$$

$\gamma_i(k)$, the factor multiplying τ_k, depends only on the allocation function, the effective spreading gains s_i', and the attenuation a_{ik}. It is hence determined by fixed system parameters and is independent of the variables p_i.

The proof of *Proposition 1* is deferred to the Appendix (see also [8]).

From the third factor in the denominator of (6), it follows that there is no feasible power allocation if there exists some base station $k \in \{1,\dots,K\}$ such that

$$\sum_{j\in\mathcal{C}(k)} \frac{1}{s_j'+1} \ge 1. \tag{7}$$

If all data rates and QoS parameters are equal, i.e., $R_j e_j = R'$, say, for all $j \in \mathcal{C}(k)$, then (7) is equivalent to $(|C(k)|-1)R' \ge \omega$. Hence, in this special case, there is no feasible power allocation whenever, for some cell, the effective data rates times the number of mobiles less one exceeds the chip rate of the system. If, like in UMTS $\omega = 3.84$ MChip/s, the data rate is 30 kb/s, and $e = 5$ dB $= 3.16$, then $R' = 94.8$, and there is no way to accommodate more than 41 users at the desired QoS requirement.

A macroscopic necessary and sufficient condition for the existence of a feasible power allocation including soft-handover, however, accepting "self-interference" in an approximate model, is given in [4].

In the following, we assume that $\sum_{j\in\mathcal{C}(k)}(1/(s_j'+1)) < 1$, for all $k=1,\dots,K$, and briefly describe how the dimensionality reduction works. With the solutions p_i from (6), $i\in\mathcal{C}(k)$, τ_k may be written as

$$\begin{aligned}\tau_k &= \tau_k^0 + \sum_{j\notin\mathcal{C}(k)} a_{jk}p_j \\ &= \tau_k^0 + \sum_{m\neq k}\sum_{j\in\mathcal{C}(m)} a_{jk}\gamma_j(m)\tau_m \\ &= \tau_k^0 + \sum_{m\neq k} c_{mk}\tau_m, \quad k=1,\dots,K\end{aligned} \tag{8}$$

with quantities $c_{mk} = \sum_{j\in\mathcal{C}(m)} a_{jk}\gamma_j(m)$, again independent of p_i.

In order to obtain a compact representation of system (8), we define the nonnegative $K\times K$ matrix $\boldsymbol{C} = (c_{mk}\bar{\delta}_{mk})_{m,k=1,\dots,K}$. $\bar{\delta}_{mk} = 1 - \delta_{mk}$ denotes the complementary Kronecker delta such that $\boldsymbol{C}$ has diagonal entries 0 and nondiagonal entries c_{mk}. Then (8) reads as

$$(\boldsymbol{I} - \boldsymbol{C}')\boldsymbol{\tau} = \boldsymbol{\tau}^0 \tag{9}$$

with the obvious notation $\boldsymbol{\tau} = (\tau_k)_{1\le k\le K}$ and $\boldsymbol{\tau}^0 = (\tau_k^0)_{1\le k\le K}$.

By Perron–Frobenius' theory, (9) has a positive solution, iff the spectral radius $\rho(\boldsymbol{C})$ satisfies $\rho(\boldsymbol{C}) < 1$, provided $\boldsymbol{C}$ is irreducible, see [12, p. 30]. This is the case, e.g., whenever $a_{ik} > 0$ for all i, k. If $\boldsymbol{C}$ is not irreducible, the problem decomposes into smaller subproblems of the same type. In summary, we have shown the following result.

Proposition 2: Assume that $\sum_{j\in\mathcal{C}(k)}(1/(s_j'+1)) < 1$, for all $k=1,\dots,K$, and that $\boldsymbol{C}$ is irreducible. Then there is a feasible power allocation $\boldsymbol{p}$ iff $\rho(\boldsymbol{C}) < 1$.

In this case, $\boldsymbol{p} = (p_i)_{1\le i\le n}$ can be efficiently computed by first solving the $K\times K$ system (9) for τ_k, and then computing $p_i = \gamma_i(k)\tau_k$, $i\in\mathcal{C}(k)$, according to (6).

The following result provides sufficient conditions for a feasible power allocation to exist or not to exist, respectively. The proof is deferred to the Appendix.

Proposition 3: Let $\zeta_1,\dots,\zeta_K > 0$ be such that $\sum_{l\neq k_i} a_{il} \le \zeta_{k_i}a_{ik_i}$, for all $i=1,\dots,n$. Then there exists a feasible power allocation if

$$\sum_{i\in\mathcal{C}(k)} \frac{1}{s_i'+1} < \frac{1}{\zeta_k+1} \quad \text{for all } k=1,\dots,K.$$

If, on the other hand, $\sum_{l\neq k_i} a_{il} \ge \zeta_{k_i}a_{ik_i}$ for all $i=1,\dots,n$, then there is no feasible power allocation if

$$\sum_{i\in\mathcal{C}(k)} \frac{1}{s_i'+1} > \frac{1}{\zeta_k+1} \quad \text{for all } k=1,\dots,K.$$

An example of how the constants $\zeta_1,\dots,\zeta_K$ could be selected is as follows. In a balanced load situation, each mobile is allocated to the base station with the least attenuated transmission, i.e., $a_{ik} \le a_{ik_i}$ for all i, k. With $\zeta_k = K-1$ the premises of *Proposition 3* are satisfied, and it follows that a feasible power allocation exists whenever

$$\sum_{i\in\mathcal{C}(k)} \frac{1}{s_i'+1} < \frac{1}{K} \quad \text{for all } k=1,\dots,K.$$

III. Capacity and Feasibility

Proposition 2 offers an efficient way to check the existence and to compute a minimal power assignment. In the following we aim at a compact representation of the set of feasible uplink user demands s_i', $i=1,\dots,n$. A vector $(s_i')_{1\le i\le n}$ is called feasible whenever there exists a minimal power setting such that E_b/N_0 for each user reaches the desired level. We still keep the assignment of mobiles to base stations fixed, in our previous notation $c: i\mapsto k_i$.

We revisit (3) in order to find a minimal global power setting $\boldsymbol{p} = (p_1,\dots,p_n)$ such that

$$s_i' \frac{p_i a_{ik_i}}{\sum_{j\neq i} p_j a_{jk_i} + \tau_{k_i}^0} = 1, \quad i=1,\dots,n \tag{10}$$

or, equivalently, after some algebra

$$p_i - \sum_{j \neq i} \frac{a_{jk_i}}{s'_i a_{ik_i}} p_j = \frac{\tau^0_{k_i}}{s'_i a_{ik_i}}, \quad i = 1, \ldots, n. \tag{11}$$

Combining the user demands s'_i in a diagonal matrix

$$\boldsymbol{S} = \operatorname{diag}(s'_1, \ldots, s'_n)$$

and using the notation (subscript UL stands for uplink)

$$\boldsymbol{B}_{\mathrm{UL}} = (b_{ij})_{i,j=1,\ldots,n}, \text{ with } b_{ij} = \begin{cases} 0, & \text{if } i = j \\ a_{jk_i}, & \text{if } i \neq j \end{cases}$$

and

$$\boldsymbol{D} = \operatorname{diag}\left(a^{-1}_{ik_i}\right)_{i=1,\ldots,n} \text{ and } \tilde{\boldsymbol{\tau}}^0 = \left(\tau^0_{k_i}\right)_{i=1,\ldots,n}$$

the system of linear equations (11) can be written as

$$(\boldsymbol{I} - \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}_{\mathrm{UL}})\boldsymbol{p} = \boldsymbol{S}^{-1}\boldsymbol{D}\tilde{\boldsymbol{\tau}}^0. \tag{12}$$

An immediate consequence of Perron–Frobenius' theory (see [12, p. 30] and [3, Th. 1]) is the following.

Proposition 4: If $\boldsymbol{B}_{\mathrm{UL}}$ is irreducible, then (11) has a unique feasible solution $\boldsymbol{p}^*$ iff $\rho(\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}_{\mathrm{UL}}) < 1$.

The effective spreading gain $s'_i = \omega/(R_i e_i)$ comprises the user requirements R_i (transmission rate) and e_i (QoS indicator) in a single term. We use the notion *capacity region* for the set of all user demands $(s'_1, \ldots, s'_n)$ which allow for an admissible power allocation. The capacity region contains all such combinations of user requirements that can be simultaneously served by the network. Its boundary points correspond via the inverse transformation $1/s'_i$ to a certain n-dimensional extension of the information-theoretic capacity.

By *Proposition 4*, the formal definition of the capacity region is given as follows:

$$\mathcal{S}_{\mathrm{UL}} = \left\{\boldsymbol{S} = \operatorname{diag}(s'_1, \ldots, s'_n) \,|\, \rho(\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}_{\mathrm{UL}}) < 1\right\}. \tag{13}$$

Obviously, $\mathcal{S}_{\mathrm{UL}}$ is independent of the background noise $\tilde{\boldsymbol{\tau}}_0$. This is due to the fact that there is no upper bound on the transmit power of mobiles. This inaccuracy of the model restricts its applicability to situations where each mobile is able to reach some base station within its transmission range.

A most exciting structural result is the following property of $\mathcal{S}_{\mathrm{UL}}$. The proof, given in the Appendix, essentially consists of showing that the spectral radius $\rho(\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}_{\mathrm{UL}})$ is logarithmically convex in the variable $\boldsymbol{S}$. In [13], it is shown by verifying the existence of some admissible power allocation that $\boldsymbol{S}_1, \boldsymbol{S}_2 \in \mathcal{S}_{\mathrm{UL}}$ entails $\boldsymbol{S}_1^{\alpha}\boldsymbol{S}_2^{(1-\alpha)} \in \mathcal{S}_{\mathrm{UL}}$ for any $\alpha \in [0,1]$, hence, the log-convexity of $\mathcal{S}_{\mathrm{UL}}$.

Proposition 5: $\mathcal{S}_{\mathrm{UL}}$ is a convex set of diagonal matrices.

The boundary points of $\mathcal{S}_{\mathrm{UL}}$ are of particular interest, since they represent the states of the system where no additional capacity can be provided. In such points, contrary interests of users must be somehow balanced according to individual utilities. Minimax and Bayes strategies are concepts to cope with conflicts of this type. In our framework, admissible strategies correspond to the boundary points of $\mathcal{S}_{\mathrm{UL}}$. Its convexity now opens the door to a widely developed theory for characterizing such strategies. Analogous concepts are used in game theory, which can also be exploited for admission control to the network in high load situations.

IV. Duality of Uplink and Downlink

It is commonly accepted that the uplink is the more restricting factor in cellular networks. In this section, we give a precise mathematical formulation of this statement in terms of capacity regions. To anticipate the main result, if the downlink capacity region $\mathcal{S}_{\mathrm{DL}}$ is defined analogously to (13) as the set of user demand vectors which allow for an admissible power control scheme, then $\mathcal{S}_{\mathrm{UL}} \subseteq \mathcal{S}_{\mathrm{DL}}$.

A fair comparison between the capacity of the uplink and downlink on the basis of system-inherent effects has to assume equal external conditions for both directions. Hence, components like allocation of mobiles and transmission gain are kept fixed for either link in what follows. To be more formal, we assume the same allocation map $c : i \mapsto k_i$. This means that user i is served by base station k_i, which allocates the amount q_i of power to the transmission to user i.

A downlink demand vector $(s'_1, \ldots, s'_n)$ for n users is feasible if there exists a componentwise positive power allocation $\boldsymbol{q} = (q_1, \ldots, q_n)$ such that

$$s'_i \frac{q_i a_{ik_i}}{\sum_{j \notin \mathcal{C}(k_i)} q_j a_{ik_j} + \sum_{j \in \mathcal{C}(k_i)\setminus\{i\}} \gamma q_j a_{ik_j} + \sigma^0_i} \geq 1 \quad \text{for all } i = 1, \ldots, n. \tag{14}$$

The numerator represents the received carrier at mobile i and the denominator combines three types of interference. The first sum collects other-cell interference, the second in-cell interference, because of code orthogonality extenuated by a factor $\gamma \in [0,1]$. Finally, σ^0_i denotes the background and thermal receiver noise at mobile station i.

Some easy algebra transforms (14) into

$$q_i - \sum_{j \neq i} \frac{\gamma_{ij} a_{ik_j}}{s'_i a_{ik_i}} q_j = \frac{\sigma^0_i}{s'_i a_{ik_i}}, \quad i = 1, \ldots, n$$

where

$$\gamma_{ij} = \begin{cases} \gamma \in [0,1], & \text{if } k_i = k_j \\ 1, & \text{otherwise.} \end{cases}$$

Resembling the step from (11) to (12), we use the notation

$$\boldsymbol{B}_{\mathrm{DL}} = (b_{ij})_{i,j=1,\ldots,n}, \quad b_{ij} = \begin{cases} 0, & \text{if } i = j \\ \gamma_{ij} a_{ik_j}, & \text{if } i \neq j \end{cases}$$

and

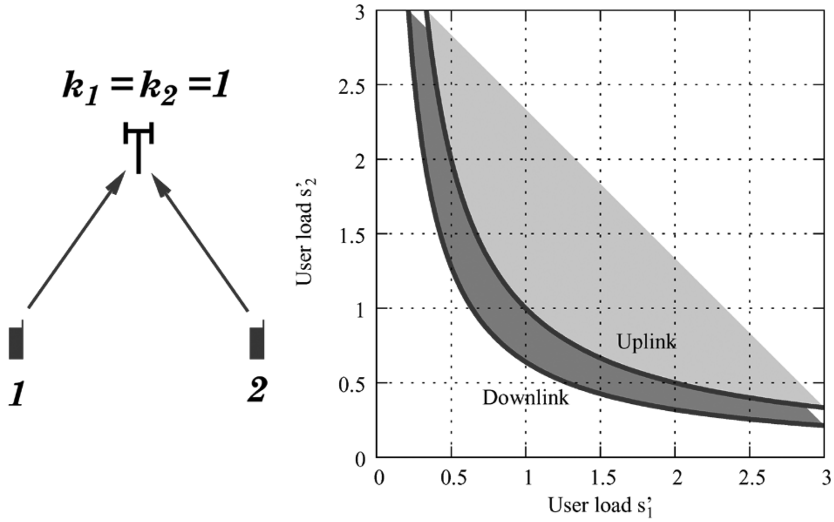
$$\boldsymbol{\sigma}^0 = \left(\sigma^0_i\right)_{i=1,\ldots,n}$$

in order to end up with the following matrix equation:

$$(\boldsymbol{I} - \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}_{\mathrm{DL}})\boldsymbol{q} = \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{\sigma}^0. \tag{15}$$

Analogously to the above, we have the following.

Proposition 6: If $\boldsymbol{B}_{\mathrm{DL}}$ is irreducible, then (15) has a unique feasible solution $\boldsymbol{q}^*$ iff $\rho(\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}_{\mathrm{DL}}) < 1$.



(a) (b)

Fig. 1. (a) Network of two mobiles and one base station. (b) Corresponding capacity regions for uplink and downlink.

In the same vein as above, the set of feasible downlink demands, or the downlink capacity region, is defined as

$$\mathcal{S}_{\mathrm{DL}} = \left\{ \boldsymbol{S} = \mathrm{diag}\,(s'_1, \ldots, s'_n) \,|\, \rho(\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}_{\mathrm{DL}}) < 1 \right\}.$$

$\mathcal{S}_{\mathrm{DL}}$ is exactly of the same structure as $\mathcal{S}_{\mathrm{UL}}$ and hence, is convex as well.

Obviously, $\boldsymbol{B}_{\mathrm{DL}} = (\gamma_{ij} a_{ik_j} \bar{\delta}_{ij}) \leq (a_{ik_j} \bar{\delta}_{ij}) = \boldsymbol{B}'_{\mathrm{UL}}$. The inequality carries over after multiplication with the same positive matrix $\boldsymbol{S}^{-1}\boldsymbol{D}$, hence yielding $\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}_{\mathrm{DL}} \leq \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}'_{\mathrm{UL}}$. Finally, from the monotonicity of the spectral radius ρ and a standard argument on transposing $\boldsymbol{B}_{\mathrm{UL}}$, it follows that $\rho(\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}_{\mathrm{DL}}) \leq \rho(\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}'_{\mathrm{UL}}) = \rho(\boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}_{\mathrm{UL}})$. This proves the central result that the downlink capacity is always higher than the uplink capacity, namely the following.

Proposition 7:

$$\mathcal{S}_{\mathrm{UL}} \subseteq \mathcal{S}_{\mathrm{DL}}.$$

The case $n = 2$ allows for a graphical representation of the sets $\mathcal{S}_{\mathrm{UL}}$ and $\mathcal{S}_{\mathrm{DL}}$. An example network of two mobiles and one base station is depicted in Fig. 1. It is straightforward to show that

$$\begin{aligned} \mathcal{S}_{\mathrm{UL}} &= \{(s'_1, s'_2) > 0 | 1 < s'_1 s'_2\} \\ \mathcal{S}_{\mathrm{DL}} &= \left\{(s'_1, s'_2) > 0 | \gamma^2 < s'_1 s'_2\right\}. \end{aligned} \tag{16}$$

If the orthogonality factor amounts to $\gamma = 0.8$, e.g., then the capacity regions are given by the shaded areas on the right-hand side of Fig. 1.

V. Soft Handover

Most of the above results carry over to certain models of soft handover. We start with downlink soft handover where each mobile receives signals simultaneously from two or more base stations. Applying maximal-ratio combining, the resulting SIR is obtained as the sum of the SIRs on each single link. Let q_{ik} denote the power assigned to mobile i by base station k. Neglecting orthogonality factors, the power control equations then read as

$$s'_i \sum_{k=1}^{K} \frac{q_{ik} a_{ik}}{\sum_{j \neq i} \sum_l q_{jl} a_{il} + \sigma_i^0} = 1, \quad i = 1, \ldots, n.$$

This setup is not directly accessible to the above methodology, however, the following simplified model allows for including soft handover into the above approach. We follow an idea of [9] and assume a fixed, *a priori* known $n \times K$ matrix $\boldsymbol{W} = (w_{ik})$, $w_{ik} \geq 0$, with $\sum_{k=1}^{K} w_{ik} = 1$. The weights w_{ik} represent the portion of the QoS parameter $1/s'_i$ for mobile i received from base station k. This approach leads to the following nK equations:

$$\frac{q_{ik} a_{ik}}{\sum_{j \neq i} \sum_l q_{jl} a_{il} + \sigma_i^0} = \frac{w_{ik}}{s'_i}, \quad i = 1, \ldots, n, \; k = 1, \ldots, K. \tag{17}$$

If $a_{ik} = 0$, then necessarily $w_{ik} = 0$ and the convention $w_{ik}/a_{ik} = 0$ is used. Introducing constants $s_{ik} = s'_i/w_{ik}$, we obtain nK linear equations in nK unknowns q_{ik} as

$$q_{ik} - \frac{1}{s'_{ik} a_{ik}} \sum_{j \neq i} \sum_{l=1}^{K} q_{jl} a_{il} = \frac{\sigma_i^0}{a_{ik} s_{ik}}, \qquad i = 1, \ldots, n, \; k = 1, \ldots, K. \tag{18}$$

With the notation $\boldsymbol{q} = (q_{11}, \ldots, q_{1K}, \ldots, q_{n1}, \ldots, q_{nK})'$, furthermore, using the $K \times K$ matrices

$$\boldsymbol{A}_i = \begin{pmatrix} a_{i1} & \cdots & a_{iK} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{iK} \end{pmatrix}$$

system (18) may be written as

$$(\boldsymbol{I} - \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B})\boldsymbol{q} = \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{\sigma}^0 \tag{19}$$

where $\boldsymbol{B}_{\mathrm{SD}}$ is blockwise composed of $\boldsymbol{A}_i$ as

$$\boldsymbol{B}_{\mathrm{SD}} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_1 \\ \boldsymbol{A}_2 & \boldsymbol{0} & \cdots & \boldsymbol{A}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_n & \boldsymbol{A}_n & \cdots & \boldsymbol{0} \end{pmatrix}$$

and the notations

$$\boldsymbol{S} = \mathrm{diag}(s_{11},\ldots,s_{1K},\ldots,s_{n1},\ldots,s_{nK})$$
$$\boldsymbol{D} = \mathrm{diag}\left(a_{11}^{-1},\ldots,a_{1K}^{-1},\ldots,a_{n1}^{-1},\ldots,a_{nK}^{-1}\right)$$
$$\boldsymbol{\sigma}^0 = \left(\sigma_1^0,\ldots,\sigma_1^0,\ldots,\sigma_n^0,\ldots,\sigma_n^0\right)'$$

are used. Unlike the different dimension, (19) and (15) are of the same structure, such that the dimensionality reduction arguments of Section II and convexity of the capacity region carry over to the soft handover model (17).

If base station receivers share the same site for supplying sectorized cells, soft handover can be applied to improve uplink capacity. Using the above linearization by w_{ik}, a general model can be developed analogously. The corresponding power allocation equations are

$$\frac{p_{ik}a_{ik}}{\sum_{j=1}^n \sum_{l=1}^K p_{jl}a_{jk} - p_{ik}a_{ik} + \tau_k^0} = \frac{w_{ik}}{s_i'}, \quad i=1,\ldots,n,\ k=1,\ldots,K \tag{20}$$

where p_{ik} denotes the power dedicated for the transmission of mobile i to base station k. Setting

$$\boldsymbol{p} = (p_{11},\ldots,p_{1K},\ldots,p_{n1},\ldots,p_{nK})'$$
$$\boldsymbol{\tau}^0 = \left(\tau_1^0,\ldots,\tau_K^0,\ldots,\tau_1^0,\ldots,\tau_K^0\right)'$$
$$\boldsymbol{B}_{\mathrm{SU}} = \begin{pmatrix} \boldsymbol{A}_1' & \cdots & \boldsymbol{A}_n' \\ \vdots & & \vdots \\ \boldsymbol{A}_1' & \cdots & \boldsymbol{A}_n' \end{pmatrix} - \boldsymbol{D}^{-1}$$

the system (20) corresponds to the matrix representation

$$(\boldsymbol{I} - \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{B}_{\mathrm{SU}})\boldsymbol{p} = \boldsymbol{S}^{-1}\boldsymbol{D}\boldsymbol{\tau}^0.$$

This equation has the same form as (12) such that *Propositions 4* and *5* carry over correspondingly to the QoS parameters s_{ik}.

VI. Log-Normal Fading

In this section, we show how the framework developed in Section III can be extended to accommodate random perturbations of the received powers. We assume log-normal variation of the received powers (shadow fading) and investigate the influence on the capacity region. Instead of one fixed capacity region $\mathcal{S}$, we consider a family of certain level-α capacity regions, $\mathcal{S}(\alpha)$, which correspond to given acceptable failure rates. It turns out that, if this rate is small, the level sets are close to being convex, cf. (21). Furthermore, we construct convex interior and exterior approximations of the level sets, which establishes a close connection to the previous results. The approximations become exact as the size of the random effects tends to zero. For definiteness, we focus on uplink capacity regions. The effects of shadow fading on the downlink regions can be dealt with in much the same way.

We assume that the channel gains are proportional to random variables of the form

$$\tilde{a}_{ik} = \beta_{ik}^{-\gamma} 10^{\frac{G_{ik}}{10}}, \quad i=1,\ldots,n;\ k=1,\ldots,K$$

where β_{ik} is the distance between mobile i and base station k, $\gamma \geq 2$ is the attenuation exponent, and the G_{ik}'s are independent normal random variables with mean zero and common variance $\sigma^2 > 0$. Experimental data suggest $\gamma = 4$ and $\sigma = 8$ (see [15]). Again, we also allow that $\tilde{a}_{ik} = 0$ if $k \neq k_i$.

The same arguments as in Section III lead us to consider the random capacity region

$$\tilde{\mathcal{S}} = \left\{\boldsymbol{S} = \mathrm{diag}(\boldsymbol{s}) | \boldsymbol{s} > \boldsymbol{0}, \rho(\boldsymbol{S}^{-1}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}) < 1\right\}$$

with random matrices $\tilde{\boldsymbol{D}} = \mathrm{diag}(\tilde{a}_{1k_1}^{-1},\ldots,\tilde{a}_{nk_n}^{-1})$ and $\tilde{\boldsymbol{B}} = (\tilde{b}_{ij})_{i,j=1}^n$, where $\tilde{b}_{ii} = 0$ and $\tilde{b}_{ij} = \tilde{a}_{jk_i}$ for $i \neq j$. It follows from *Proposition 5* that, with probability one, the random set $\tilde{\mathcal{S}}$ is a convex set. Due to the difficult analytical behavior of the spectral radius, it is, except for trivial cases, presumably impracticable to determine the distributions of the random set $\tilde{\mathcal{S}}$ and of $\rho(\boldsymbol{S}^{-1}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}})$ for any fixed $\boldsymbol{S}$.

In the deterministic case, one is interested in whether a given $\boldsymbol{S}$ belongs to $\mathcal{S}$. In the stochastic case, interest shifts toward describing the demand profiles that can be served with a prescribed probability close to one. Thus, for $0 < \alpha < 1$, we consider the deterministic *level-α capacity region*

$$\mathcal{S}(\alpha) = \left\{\boldsymbol{S} | P[\boldsymbol{S} \in \tilde{\mathcal{S}}] \geq 1-\alpha\right\} = \left\{\boldsymbol{S} | P\left[\rho(\boldsymbol{S}^{-1}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}) < 1\right] \geq 1-\alpha\right\}.$$

Obviously, these level sets are increasing in α, and $\mathcal{S}(\alpha) \nearrow \{\boldsymbol{S} = \mathrm{diag}(\boldsymbol{s}) | \boldsymbol{s} > \boldsymbol{0}\}$ as $\alpha \nearrow 1$, and $\mathcal{S}(\alpha) \searrow \emptyset$ as $\alpha \searrow 0$. If $\boldsymbol{S}_1 \in \mathcal{S}(\alpha)$ and $\boldsymbol{S}_2 \geq \boldsymbol{S}_1$, then $\boldsymbol{S}_2 \in \mathcal{S}(\alpha)$. This is a consequence of the monotonicity of the spectral radius (see [7, p. 491]). Moreover

$$\boldsymbol{S}_1, \boldsymbol{S}_2 \in \mathcal{S}(\alpha) \implies \lambda\boldsymbol{S}_1 + (1-\lambda)\boldsymbol{S}_2 \in \mathcal{S}(\alpha') \quad \text{for all } 0 \leq \lambda \leq 1, \alpha' \geq 2\alpha. \tag{21}$$

For, if $\boldsymbol{S}_1, \boldsymbol{S}_2 \in \mathcal{S}(\alpha)$, then

$$P\left[\max\left\{\rho\left(\boldsymbol{S}_1^{-1}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}\right), \rho\left(\boldsymbol{S}_2^{-1}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}\right)\right\} < 1\right] > 1-2\alpha$$

and it follows from *Proposition 5* that

$$P\left[\rho\left([\lambda\boldsymbol{S}_1 + (1-\lambda)\boldsymbol{S}_2]^{-1}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}\right) < 1 \text{ for all } \lambda\right] \geq 1-2\alpha.$$

In the case of $n = 2$ mobiles, each mobile $i \in \{1,2\}$ connected to a different base station k_1 and k_2, the level-α capacity region can be explicitly determined. After some algebra, it follows that

$$\mathcal{S}(\alpha) = \{(s_1', s_2') > 0 | P\left(\frac{\beta_{1k_2}^{-\gamma}}{\beta_{1k_1}^{-\gamma}} \frac{\beta_{2k_1}^{-\gamma}}{\beta_{2k_2}^{-\mu}} 10^{\frac{G_{1k_2}+G_{2k_1}-G_{1k_1}-G_{2k_2}}{10}} < s_1's_2'\right) \geq 1-\alpha\}.$$

Abbreviating the distance-dependent term by c and $G = G_{1k_2} + G_{2k_1} - G_{1k_1} - G_{2k_2}$, we obtain

$$\mathcal{S}(\alpha) = \left\{(s_1', s_2') > 0 | P\left(c10^{\frac{G}{10}} < s_1's_2'\right) \geq 1-\alpha\right\}$$

where G is normally distributed with zero mean and variance $4\sigma^2$. Easy probability calculus yields the following description:

$$\mathcal{S}(\alpha) = \left\{(s_1', s_2') > 0 | s_1's_2' > c10^{\frac{\sigma u_{1-\alpha}}{5}}\right\}$$

of the level-α capacity region, where $u_{1-\alpha}$ is the $(1-\alpha)$-fractile of the standard normal distribution. Thus, $\mathcal{S}(\alpha)$ is essentially of the same form as (16) with an additional factor $10^{\sigma u_{1-\alpha}/5}$.

To gain insight into the geometric properties of the level sets $\mathcal{S}(\alpha)$ for arbitrary $n \geq 2$, consider the sets

$$\mathcal{T}(\lambda) = \left\{ \boldsymbol{S} = \mathrm{diag}(\boldsymbol{s}) | \rho\left(\boldsymbol{S}^{-1} E[\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}]\right) < \lambda \right\}, \quad \lambda > 0.$$

In the deterministic case, $\sigma = 0$ and $\mathcal{S} = \mathcal{T}(\lambda)$ with $\lambda = 1$. Thus, $\mathcal{T}(1)$ is convex, and the log-convexity of the spectral radius, established in the Appendix, shows that $\mathcal{T}(\lambda)$ is, in fact, convex for every $\lambda > 0$ and $\sigma > 0$. There seems to be no reasonable interpretation of the sets $\mathcal{T}(\lambda)$ with $\lambda \neq 1$ in the deterministic setting. In the presence of shadow fading, the sets occur quite naturally in approximating the level sets $\mathcal{S}(\alpha)$. Note that the sets $\mathcal{T}(\lambda)$ form an increasing family of convex sets, and $\mathcal{T}(\lambda) \nearrow \{\boldsymbol{S} = \mathrm{diag}(\boldsymbol{s}) | \boldsymbol{s} > \boldsymbol{0}\}$ as $\lambda \nearrow \infty$.

To relate the $\mathcal{T}(\lambda)$ to the level-α capacity regions, observe first that, if $X \sim \mathcal{N}(0, \sigma^2)$, then

$$E[10^X] = 10^{\frac{\sigma^2}{2} \log 10}$$
$$P\left\{10^X \leq \lambda E[10^X]\right\} = \Phi\left[\frac{\log \lambda}{\sigma \log 10} + \frac{\sigma}{2} \log 10\right]$$

where Φ is the standard normal distribution function. Clearly, if $i = j$ or $\tilde{a}_{jk_i} = 0$, then $E\{\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}\}_{ij} = 0$. Otherwise, it follows that

$$\{\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}\}_{ij} = \frac{\tilde{a}_{jk_i}}{\tilde{a}_{ik_i}} = \frac{\beta_{jk_i}^{-\gamma}}{\beta_{ik_i}^{-\gamma}} 10^{\frac{G_{jk_i} - G_{ik_i}}{10}}$$
$$E\{\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}\}_{ij} = \frac{\beta_{jk_i}^{-\gamma}}{\beta_{ik_i}^{-\gamma}} 10^{\frac{\sigma^2 \log 10}{100}}$$

and, for every $\lambda > 0$

$$\begin{aligned} P&\left[\{\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}\}_{ij} \leq \lambda E\{\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}\}_{ij}\right] \\ &= P\left[10^{\frac{G_{jk_i} - G_{ik_i}}{10}} \leq \lambda 10^{\frac{\sigma^2 \log 10}{100}}\right] \\ &= \nu(\lambda, \sigma) \end{aligned}$$

where

$$\nu(\lambda, \sigma) = \Phi\left[\frac{10 \log \lambda}{\sqrt{2}\sigma \log 10} + \frac{\sigma \log 10}{10\sqrt{2}}\right].$$

Let κ denote the number of pairs i, j for which $E\{\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}\}_{ij} > 0$. Notice that the larger κ is, the more convoluted the CDMA system is. In any case, $\kappa \leq n(n-1)$. To avoid trivialities, assume $\kappa > 1$. Applying the inequality $P(\cap E_i) \geq 1 - \sum P(E_i^C)$, one obtains

$$\begin{aligned} P\left[\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}} \leq \lambda E[\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}]\right] &\geq 1 - \kappa\left[1 - \nu(\lambda, \sigma)\right] \\ P\left[\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}} \geq \lambda E[\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}]\right] &\geq 1 - \kappa\nu(\lambda, \sigma). \end{aligned} \tag{22}$$

For every fixed σ, $\nu(\lambda, \sigma)$ is strictly increasing in $\lambda > 0$, and

$$\lim_{\lambda \to 0} \nu(\lambda, \sigma) = 0, \quad \lim_{\lambda \to \infty} \nu(\lambda, \sigma) = 1.$$

Proposition 8: Fix $0 < \alpha < 1$. Let $\lambda_1 = \lambda_1(\sigma) > 0$, $\lambda_2 = \lambda_2(\sigma) > 0$ be given by

$$\nu(\lambda_1, \sigma) = \frac{1-\alpha}{\kappa}, \quad \nu(\lambda_2, \sigma) = 1 - \frac{\alpha}{\kappa}.$$

Then $\lambda_1 < \lambda_2$, and

$$\mathcal{T}\left(\lambda_2^{-1}\right) \subset \mathcal{S}(\alpha) \subset \mathcal{T}\left(\lambda_1^{-1}\right).$$

Moreover

$$\lim_{\sigma \to 0} \lambda_1(\sigma) = \lim_{\sigma \to 0} \lambda_2(\sigma) = 1.$$

To see this, let $\boldsymbol{S} \in \mathcal{T}(\lambda_2^{-1})$. That is, $\lambda_2 \rho(\boldsymbol{S}^{-1} E[\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}]) < 1$. From this, the monotonicity of the spectral radius, and (22), it follows that

$$\begin{aligned} P&\left\{\rho(\boldsymbol{S}^{-1}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}) < 1\right\} \\ &\geq P\left\{\rho(\boldsymbol{S}^{-1}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}) \leq \lambda_2 \rho\left(\boldsymbol{S}^{-1} E[\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}]\right)\right\} \\ &\geq P\left\{\boldsymbol{S}^{-1}\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}} \leq \lambda_2 \boldsymbol{S}^{-1} E[\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}]\right\} \\ &= P\left\{\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}} \leq \lambda_2 E[\tilde{\boldsymbol{D}}\tilde{\boldsymbol{B}}]\right\} \\ &\geq 1 - \kappa\left[1 - \nu(\lambda_2, \sigma)\right] \\ &= 1 - \alpha. \end{aligned}$$

Hence, $\boldsymbol{S} \in \mathcal{S}(\alpha)$, so that $\mathcal{T}(\lambda_2^{-1}) \subset \mathcal{S}(\alpha)$. The proof of the other inclusion is similar. The limit assertions are easily verified.

Proposition 8 states that the level-α capacity region $\mathcal{S}(\alpha)$ can be sandwiched between two convex sets which come arbitrarily close as σ becomes small. We conjecture that $\mathcal{S}(\alpha)$ itself is a convex set for any $0 \leq \alpha \leq 1$.

VII. Conclusion and Outlook

In this paper, we developed an efficient algorithm for solving the power control problem offline. Nice and intuitive conditions on the existence of feasible power allocation schemes were given in terms of the effective spreading gains or data rates, respectively. We introduced the uplink and downlink capacity region of cellular CDMA networks by bounding the spectral radius of a matrix built of system and transmission gain parameters. Intuitively speaking, the regions represent the user demands which can be carried by the network. As a structural result, it was shown that the uplink capacity region is a subset of the downlink capacity region and that both are convex n-dimensional sets. We also show that our methods carry over when softhandover is described by a linearized model which assumes *a priori* knowledge of the portions combining the received signal.

Convexity is particularly important for access control strategies via pricing the data rates and quality demands of users in an equitable way. In the framework of game theory, Nash equilibria and monotone and Pareto solutions will provide the means to balance conflicting interests of users. Moreover, the boundary points of the capacity region correspond to admissible strategies in the framework of Bayes and min–max decision rules which will be exploited for future investigations.

The basic model assumes fixed transmission gains, hence taking a snapshot of an in-reality highly dynamic scenario. We introduce the concept of level-α capacity regions when transmission gains are subject to log-normal shadowing. In this case, an interesting sandwich property is derived, where both the upper and lower framing set are convex.

Future work will be devoted to further exploiting the rich structure of the capacity region for the design of admission control policies. We also aim at a more comprehensive approach for dealing with the capacity region in case of random fading.

APPENDIX

Proof of Proposition 1

System (5) is extended by the equation $\sum_{j\in\mathcal{C}(k)} a_{jk}p_j = c$ with an additional variable c. Substituting this into (5) gives

$$a_{ik}p_i = \frac{c+\tau_k}{s'_i+1} \text{ and } c = \sum_{j\in\mathcal{C}(k)} a_{jk}p_j = (c+\tau_k)\sum_{j\in\mathcal{C}(k)} \frac{1}{s'_j+1}.$$

Solving the last equation for c yields $c = \tau_k\beta/(1-\beta)$ with $\beta = \sum_{j\in\mathcal{C}(k)} 1/(s'_j+1)$. Hence

$$p_i = \frac{\tau_k}{a_{ik}(s'_i+1)}\left(\frac{\beta}{1-\beta}+1\right) = \frac{\tau_k}{a_{ik}(s'_i+1)(1-\beta)}$$

which completes the proof. ∎

Proof of Proposition 3

Fix $m \in \{1,\ldots,K\}$, and consider the mth row sum of $\boldsymbol{C} = (c_{mk}\bar{\delta}_{mk})$ as follows:

$$\begin{aligned}\sum_{k\neq m} c_{mk} &= \sum_{k\neq m}\sum_{j\in\mathcal{C}(m)} a_{jk}\gamma_j(m)\\ &= \sum_{j\in\mathcal{C}(m)} \frac{1}{(s'_j+1)\left(1-\sum_{l\in\mathcal{C}(m)}\frac{1}{s'_l+1}\right)}\sum_{k\neq m}\frac{a_{jk}}{a_{jm}}\\ &\le \zeta_m \sum_{j\in\mathcal{C}(m)} \frac{1}{(s'_j+1)\left(1-\sum_{l\in\mathcal{C}(m)}\frac{1}{s'_l+1}\right)}\\ &= \zeta_m \frac{\sum_{l\in\mathcal{C}(m)}\frac{1}{s'_l+1}}{1-\sum_{l\in\mathcal{C}(m)}\frac{1}{s'_l+1}}.\end{aligned}$$

The right-hand side of this chain is <1 if and only if

$$\sum_{l\in\mathcal{C}(m)}\frac{1}{s'_l+1} < \frac{1}{\zeta_m+1}. \tag{23}$$

Hence, if (23) is satisfied for all $m = 1,\ldots K$, the assertion follows from the fact that the spectral radius satisfies $\rho(\mathbf{C}) \le \max_m \sum_{k\neq m} c_{mk}$, see [7, p. 492]. Reverting the above inequality signs yields the second assertion along the same lines, since $\rho(\mathbf{C}) \ge \min_m \sum_{k\neq m} c_{mk}$. ∎

Proof of Proposition 5

The concept of log-convex functions turns out to be an essential ingredient to the proof. A positive function f is called log-convex if $\log f$ is convex. The following properties are well known, see [10, p. 19].

1) If f and g are log-convex functions, then $f+g$ and $f\cdot g$ are log-convex.
2) If f is a log-convex function, and $\alpha > 0$, then f^α is log-convex.
3) For any convergent sequence f_n of log-convex functions, the limit $f = \lim_{m\to\infty} f_m$ is log-convex as well, provided the limit is strictly positive.

Now, slightly simplifying the notation, we have to show that

$$\mathcal{S} = \left\{\boldsymbol{S} = \operatorname{diag}(s_1,\ldots,s_n) | \rho(\boldsymbol{S}^{-1}\boldsymbol{B}) < 1, s_i > 0, i = 1,\ldots,n\right\}$$

is convex for any nonnegative matrix $\boldsymbol{B}$.

We first treat the case that $\boldsymbol{B}$ has positive entries b_{ij} and show that

$$\rho(\boldsymbol{A}(\lambda)) = \rho\left((\lambda\boldsymbol{S}_1 + (1-\lambda)\boldsymbol{S}_2)^{-1}\boldsymbol{B}\right) < 1$$

for all $\lambda \in [0,1]$, $\boldsymbol{S}_1 = \operatorname{diag}(s_i^{(1)})$, $\boldsymbol{S}_2 = \operatorname{diag}(s_i^{(2)}) \in \mathcal{S}$.

For this purpose, we use the representation (see [7, p. 299]) $\rho(\mathbf{C}) = \lim_{k\to\infty} \|\mathbf{C}^k\|_1^{1/k}$, where $\|\mathbf{C}\|_1 = \sum_{ij}|c_{ij}|$ denotes the ℓ_1 matrix norm.

The entries of $\boldsymbol{A}(\lambda)$

$$a_{ij}(\lambda) = \frac{b_{ij}}{\lambda s_i^{(1)} + (1-\lambda)s_i^{(2)}}$$

are log-convex functions on [0,1], since $x \mapsto 1/x$ is log-convex on $(0, \infty)$. By definition of $\|\cdot\|_1$, using 1) iteratively yields that $\|\boldsymbol{A}(\lambda)^k\|_1$ is log-convex. By 2), $\|\boldsymbol{A}(\lambda)^k\|_1^{1/k}$ is log-convex, and by 3), the same holds for the limit $\lim_{k\to\infty}\|\boldsymbol{A}(\lambda)^k\|_1^{1/k} = \rho(\boldsymbol{A}(\lambda))$. Note that $\boldsymbol{B} > 0$ componentwise ensures that $\rho(\boldsymbol{A}(\lambda)) > 0$, cf. [7, p. 496]. Hence

$$\begin{aligned}\rho(\boldsymbol{A}(\lambda)) &\le \max\{\rho(\boldsymbol{A}(0)), \rho(\boldsymbol{A}(1))\}\\ &= \max\left\{\rho\left(\boldsymbol{S}_2^{-1}\boldsymbol{B}\right), \rho\left(\boldsymbol{S}_1^{-1}\boldsymbol{B}\right)\right\} < 1\end{aligned}$$

which shows the assertion for positive $\boldsymbol{B}$.

If $\boldsymbol{B}$ is merely nonnegative, consider the positive matrices $\boldsymbol{B}_k = \boldsymbol{B} + (1/k)\mathbf{1}_{n\times n}$ and the convex sets

$$\begin{aligned}\mathcal{S}_k = \Big\{\boldsymbol{S} = \operatorname{diag}(s_1,\ldots,s_n) | \rho(\boldsymbol{S}^{-1}\boldsymbol{B}_k) < 1,\\ s_i > 0, i = 1,\ldots,n\}, \quad k \in \mathbb{N}.\end{aligned}$$

Since, for every $\boldsymbol{S}$, $\rho(\boldsymbol{S}^{-1}\boldsymbol{B}_{k+1}) \le \rho(\boldsymbol{S}^{-1}\boldsymbol{B}_k)$ and $\lim_{k\to\infty}\rho(\boldsymbol{S}^{-1}\boldsymbol{B}_k) = \rho(\boldsymbol{S}^{-1}\boldsymbol{B})$, the sets $\mathcal{S}_k$ form an increasing sequence of convex sets with $\mathcal{S} = \bigcup_{k=1}^\infty \mathcal{S}_k$. It follows that $\mathcal{S}$ is convex too, which completes the proof. ∎

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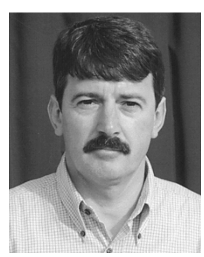

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