Optimal and Efficient Bit Loading for OFDM in the Presence of Channel Uncertainty

Chunhui Liu [1], Rudolf Mathar [2]

Institute for Theoretical Information Technology, RWTH Aachen University
D-52056 Aachen, Germany
[1] liu@ti.rwth-aachen.de
[2] mathar@ti.rwth-aachen.de

***Abstract*—In practice only imperfect channel state information (CSI) is available at the transmitter due to noisy and time-varying channel. In this paper, we consider optimal bit loading for orthogonal frequency division multiplexing (OFDM) with imperfect CSI subject to channel estimation. The frame length comes in as an important control parameter, while soft channel estimation and the Cramer-Rao bound are used to include the effect of channel estimation. We modify the conventional water-filling to take the maximal allowed rate on each subcarrier into account while considering CSI error. Its continuous output rates are optimally quantized by a non-iterative algorithm. Simulations show that the frame length is an important system parameter and that the computing time for the proposed algorithm linearly increases with the increasing number of subcarriers N.**

I. Introduction

In OFDM different modulation schemes can be employed on subcarriers in order to adapt to individual channel gains on subcarriers. To take advantages of adaptive modulation accurate CSI is required at the transmitter. However, only imperfect CSI is available in practice due to noisy channel estimation and channel feedback delay. The induced performance degradation has been studied in [1], [2].

Although the margin maximization problem (MMP) has been solved by water-filling [3] with continuous rates allowed, water-filling has to be modified when rates are constrained to be discrete. Algorithms for optimal bit loading suggested in [4], [5] get the complexity of around $\mathcal{O}(N\log(N))$. The algorithm given in [6] has low complexity but cannot provide the optimal solution.

In this paper, we quantify the performance degradation of bit loading for OFDM by using soft channel estimates, which additionally include channel uncertainties [7]. With the Cramer-Rao lower bound (CRLB) on the variance of channel estimation error we derive results not depending on a specific estimation method. Furthermore, an algorithm is proposed to implement bit loading for OFDM. It has the complexity of $\mathcal{O}(N)$ the same as the algorithms suggested in [8], [9].

The remainder of this paper is organized as follows. Section II presents the system model. In Section III effective noise is embedded with channel estimation error. In Section IV we analyze the influence of channel estimation error on water-filling for OFDM. An algorithm, quantizing the continuous output rates, is explained in Section V. Numerical results are given in Section VI. Finally this paper is concluded.

II. System Model and Assumptions

Consider an OFDM system with N subcarriers. The data stream is divided into frames. Each frame consists of I OFDM symbols and may include I_t pilot symbols or not. It is assumed that transmission is subject to frequency selective fading and that the channel is invariant within a sufficiently long period of time. In frequency domain a received symbol on the nth subcarrier may be written as

$$y[n] = h[n]x[n] + \omega[n], \tag{1}$$

where $y[n]$ and $x[n]$ refer to the received and transmitted symbols on the nth subcarrier, respectively. Additive noise sample $\omega[n]$ is complex Gaussian distributed with zero mean and variance σ_ω^2. Let $h[n]$ denote the channel coefficient on the nth subcarrier in frequency domain.

In practice, reliable transmission always requires low bit-error rates (BERs), see [10]. Therefore, we assume our systems throughout this paper to be operated in the high signal-to-noise ratio (SNR) range, where low BERs can be guaranteed.

III. Imperfect CSI by Noisy Channel Estimation

By assuming that the channel remains invariant for a sufficiently long period of time, channel feedback delay may be neglected so that we can focus on the effect of the imperfect CSI only subject to channel estimation.

Channel estimation error in OFDM systems may be measured by the average mean square error (MSE) over subcarriers, defined as

$$\text{MSE} = E\left\{\frac{1}{N}\sum_{n=1}^{N}|h[n]-\hat{h}[n]|^2\right\},$$

where $\hat{h}[n]$ denotes the estimated channel coefficient on the nth subcarrier. If multipath components experience independent Rayleigh fading, it follows that all subcarriers undergo identical Rayleigh fading [1]. The MSE may be written as

$$\text{MSE} = E\left\{|h[n]-\hat{h}[n]|^2\right\}, \forall\, n \in \{1,\ldots,N\}.$$

For simplicity, the subcarrier index is suppressed within this section. With the least-squares channel estimation, the estimated channel coefficient can be

$$\hat{h} = \frac{y}{x} = h + \underbrace{\frac{\omega}{x}}_{e}.$$

Obviously, h can be expressed by

$$h = \hat{h} - e.$$

If x is known or correctly decided by the receiver, channel estimation error e is a zero-mean complex Gaussian random variable with variance $\sigma_e^2 = \sigma_\omega^2/|x|^2$. Therefore, h can be treated as a complex Gaussian random variable with mean $\hat{h}$ and variance σ_e^2. In such a case it holds that $\text{MSE} = \sigma_e^2$.

Hence, σ_e^2 may be interpreted as the uncertainty of h. The pair $(\hat{h}, \sigma_e^2)$ is called a soft channel estimate, it extends the channel estimate $\hat{h}$ by the channel uncertainty σ_e^2, see [7].

Based on the decomposition above, (1) can be re-written as

$$y = \hat{h}x \underbrace{-ex + \omega}_{\eta}.$$

The term $\eta = -ex + \omega$ is called effective noise. It is comprised of additive channel noise and channel estimation error. Since ω is independent over OFDM symbols and then e can be assumed to be stochastically independent, it easily follows that

$$\sigma_\eta^2 = E\{|-ex + \omega|^2\} = P\sigma_e^2 + \sigma_\omega^2,$$

where $P = E\{|x|^2\}$ is the transmit power of data symbols.

Channel estimation in OFDM systems has already been studied over many years. There are essentially two types of methods: the pilot-based channel estimation and the blind or semi-blind channel estimation.

The first approach is based only on pilot symbols, where the MSE can be expressed by the CRLB in [11], given as

$$\overline{\text{MSE}} = \frac{c}{I_t \text{SNR}}, \tag{2}$$

where $c = L/N$ in [12] and L is the length of channel impulse response. MSE is normalized by $E\{|h|^2\}$ as $\overline{\text{MSE}}$.

The second method additionally utilizes data symbols. Channel estimation and data detection are jointly performed, mutually benefiting from each other. Its MSE is close to the CRLB at high SNR with low BERs assured as assumed earlier.

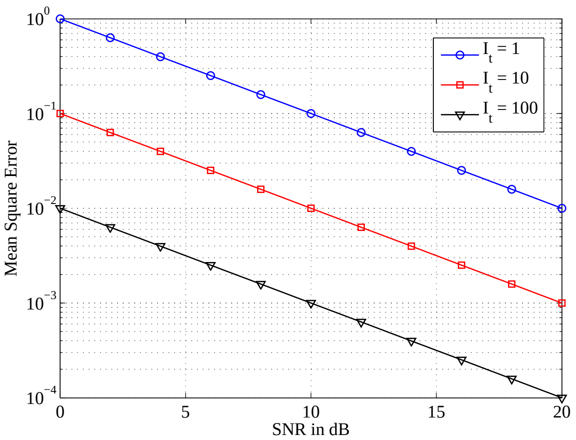

Fig. 1. The Cramer-Rao lower bound for the mean square error of channel estimation when $I_t = 1$, $I_t = 10$ and $I_t = 100$ with c = 1.

In such a case I_t in (2) is equal to the frame length I. Fig. 1 plots the CRLB against the SNR for different values of I_t.

If the pilot-based channel estimation is applied and the same transmit power is allocated on data and pilot symbols , or if the blind channel estimation is adopted, the power of η is

$$\sigma_\eta^2 = \sigma_\omega^2 (1 + \frac{c}{I_t}). \tag{3}$$

It is only impacted by the frame length when σ_ω^2 is constant.

IV. Water-Filling with Imperfect CSI

In adaptive OFDM systems the MMP is equivalent to minimizing transmit power under the BER and data rate constraints. In mathematical terms it reads as

$$\min \sum_{n=1}^{N} P[n] \tag{4}$$

subject to :

$$\begin{aligned} C1: &\quad \sum_{n=1}^{N} r[n] \geq R \\ C2: &\quad 0 \leq r[n] \leq M, \ \forall n \in \{1, \dots, N\} \\ C3: &\quad r[n] = \log_2(1 + P[n]G[n]) \end{aligned}$$

where R in $C1$ is the minimal required data rate. The constraint on the maximal allowed rate over each subcarrier M refers to the maximal amount of bits that can be transmitted over each subcarrier in one OFDM symbol. Constraint $C3$ is the power-rate function, where $P[n]$ and $r[n]$ denote the power and rate allocated on the nth subcarrier, respectively.

With *perfect* CSI the channel-to-noise ratio (CNR) is

$$G[n] = \frac{|h[n]|^2}{\Gamma \sigma_\omega^2}, \tag{5}$$

where the SNR gap Γ is a function of the required BER. With *imperfect* CSI, the effective noise (3) can be employed and the CNR becomes

$$G[n] = \frac{|\hat{h}[n]|^2}{\Gamma \sigma_\omega^2 (1 + \frac{c}{I_t})}. \tag{6}$$

The optimal power and rate allocation on each subcarrier can be derived by water-filling [3], shown as

$$P[n] = \lambda - \frac{1}{G[n]} \tag{7}$$

$$r[n] = \log_2(\lambda G[n]), \tag{8}$$

where

$$\lambda = 2^{\frac{R}{d}} \left(\prod_{n \in \mathcal{D}} \frac{1}{G[n]} \right)^{\frac{1}{d}} \tag{9}$$

is the water level determined by the rate constraint. Set $\mathcal{D}$ contains d used subcarriers. By taking (5) and (6) into (7), (8) and (9) separately, to satisfy the rate demand more power is allocated on each subcarrier with imperfect CSI compared to the one with perfect CSI. Such impact is only affected by I_t.

Algorithm 1 Strict Water-Filling (SWF)

initialization
  $\mathcal{D} \leftarrow \{1, \ldots, N\}$
  $\mathcal{A} \leftarrow \emptyset$
  $P_M[n] = \frac{1}{G[n]}(2^M - 1), n \in \mathcal{D}$
repeat
  $\lambda \leftarrow$(9)
  $P[n] \leftarrow (7), n \in \mathcal{D}$
  $\mathcal{S} \leftarrow \{n \in \mathcal{D} \mid P[n] \leq 0\}$
  $\mathcal{L} \leftarrow \{n \in \mathcal{D} \mid P[n] \geq P_M[n]\}$
  if $\mathcal{S} \neq \emptyset$ **then**
    $\mathcal{D} \leftarrow \mathcal{D} \setminus \mathcal{S}$
  else if $\mathcal{L} \neq \emptyset$ **then**
    $\mathcal{A} \leftarrow \mathcal{A} \cup \mathcal{L}$
    $\mathcal{D} \leftarrow \{1, \ldots, N\} \setminus \mathcal{A}$
    $R \leftarrow R - M \times |\mathcal{L}|$
  end if
until $\mathcal{S} = \emptyset$ and $\mathcal{L} = \emptyset$
$r[n] \leftarrow (8), n \in \mathcal{D}$
Output
  $r[n]$, λ, $\mathcal{D}$ and $\mathcal{A}$

Algorithm 1 returns the optimal solution to MMP and is called strict water-filling (SWF), because the constraint on the maximal allowed rate over each subcarrier is met unlike the continuous rate distribution in [3], [4], [9], [13].

The subcarriers, achieving the maximal allowed rate, are included in set $\mathcal{A}$. In practice, due to large rate demands most of the subcarriers are used, so $\mathcal{D}$ is initialized as containing all available subcarriers. In each iteration, if negative rates happen on subcarriers in $\mathcal{S}$, this set of subcarriers must be removed from $\mathcal{D}$. After such removing the water level and the rates on other subcarriers in $\mathcal{D}$ decrease and some subcarriers may be excluded from $\mathcal{L}$, which contains the subcarriers with rates larger than or equal to the maximal allowed rate.

Hence, we can move the subcarriers with rates larger than or equal to M bits per OFDM symbol from $\mathcal{D}$ to $\mathcal{A}$, provided there arise only positive rates on other subcarriers in $\mathcal{D}$. After that the water level increases and subcarriers, which used to achieve non-positive rates, may load positive rates. Then $\mathcal{D}$ must be set to contain all subcarriers except the ones in $\mathcal{A}$ to investigate the possibility of using the before removed subcarriers. SWF finishes till that all subcarriers in $\mathcal{D}$ achieve positive rates smaller than M bits per OFDM symbol. It can be expressed by

$$(r[n], \lambda, \mathcal{D}, \mathcal{A}) = \text{SWF}(R, M), \ \forall n.$$

The complexity of SWF depends on d, which is determined by R, N and CNRs. To simplify this matter, we assume that $d/2$ subcarriers are removed from $\mathcal{D}$ in each iteration. In such a case, N multiply operations to have $P_M[n]$, N log operations to derive $r[n]$ and maximal $2N$ multiply operations plus $\log_2(N)$ exponential operations to obtain λ must be executed. Besides these, only simple operations are needed. Hence, the complexity of SWF is around $\mathcal{O}(N)$.

V. Rate Quantization

The rates, distributed on subcarriers by SWF, can be any real numbers in $[0, M]$. However, in practical transmission, only discrete rate distribution is realizable, where fractional rates may be feasible by channel coding.

Let β denote the granularity for the modulation schemes used in the considered OFDM system. It describes the constant distance between two neighboring rates.

Rounding up a continuous rate to a nearest available discrete rate is expressed as

$$r_{\text{Q}}^{+}[n] = r[n] + \Delta r^{+}[n]$$

and rounding down a continuous rate to a nearest available discrete rate is expressed as

$$r_{\text{Q}}^{-}[n] = r[n] - \Delta r^{-}[n],$$

where $\Delta r^{+}[n], \Delta r^{-}[n] \in [0, \beta)$ denote the bit increment and decrement by rounding up and down, respectively. Accordingly, with the inverse of the power-rate function the power increment on each subcarrier can be expressed by

$$\Delta P^{+}[n] \quad = \quad \frac{1}{G[n]} 2^{r[n]}(2^{\Delta r^{+}[n]} - 1).$$

Assume that the rates on the lth and nth subcarriers, given by SWF, are positive with $\Delta r^{+}[l] > \Delta r^{+}[n]$. It follows that the power increment on the lth subcarrier must be larger than the one on the nth subcarrier, shown as

$$\begin{aligned} \Delta P^{+}[l] &= \lambda(2^{\Delta r^{+}[l]} - 1) \\ &> \lambda(2^{\Delta r^{+}[n]} - 1) = \Delta P^{+}[n] \end{aligned}$$

due to (8) and that the exponential function is monotonically increasing. This can be concluded as follows.

Theorem 1: Given the continuous rates on two subcarriers by SWF, the less bit increment results in a lower power increment than the larger one.

Recall that the number of bits per modulation symbol consecutively increases by step size β for all available modulation schemes. After the SWF, the rates on some subcarriers must be rounded up, while the rates on the others must be rounded down, so that the data rate constraint $C1$ can be met due to that it holds

$$\begin{aligned} d &> \frac{1}{\beta} \sum_{n \in \mathcal{D}} \Delta r^{+}[n] \quad \text{and} \\ d &> \frac{1}{\beta} \sum_{n \in \mathcal{D}} \Delta r^{-}[n]. \end{aligned}$$

Obviously, if a rate on one subcarrier is rounded up by $\Delta r^{+}[l] \geq \beta$, there must exist subcarriers instead, on which rates may be rounded up by bit increments smaller than β, resulting in a lower power increment while satisfying the data rate requirements.

Based on the analysis above, it can be deduced that a continuous rate, given by SWF, can be rounded up only by smaller than β bits and that subcarriers with zero rates from SWF cannot load any bits after the rate quantization.

Algorithm 2 Efficient Bit Loading (EBL)

initialization

$(r[n], \lambda, \mathcal{D}, \mathcal{A}) \leftarrow \text{SWF}(R, M), \ \forall\, n$

quantization

$R_r \leftarrow \sum_{n\in\mathcal{D}} \Delta r^-[n]$

$\mathcal{B} \leftarrow \{\lceil R_r/\beta \rceil$ subcarriers in $\mathcal{D}$ with smallest $\Delta r^+[n]\}$

$r[n] \leftarrow M, \ n \in \mathcal{A}$

$r[n] \leftarrow r_{\mathrm{Q}}^+[n], \ n \in \mathcal{B}$

$r[n] \leftarrow r_{\mathrm{Q}}^-[n], \ n \in \mathcal{D} \setminus \mathcal{B}$

$P[n] \leftarrow$ the inverse of $C3$

Output

$r[n], P[n]$

With the above theorem and deduction, a non-iterative rate quantization method for optimal bit loading is designed, shown in Algorithm 2. Algorithm 1 is used to initialize the rate distribution on subcarriers, where continuous rates are allowed and the constraint on the maximal allowed rate over each subcarrier is met. The unachieved rate R_r after rounding rates down is calculated. Then the rounding up has to be performed to meet the rate requirement R_Q again. The number of subcarriers to be increased by rate β is obviously given by $\lceil \frac{R_r}{\beta} \rceil$. Using Theorem 1 we choose the set $\mathcal{B}$ with $\lceil \frac{R_r}{\beta} \rceil$ elements containing the subcarriers $n \in \mathcal{D}$ with the smallest $\Delta r^+[n]$. Finally the rates on these subcarriers are rounded up and transmit power $P[n]$ on each used subcarrier is calculated. In the quantization step only simple operations, like plus, minus and compare are required.

In [4], [5], [13], [14], after rounding the continuous rates with a ceiling rate M, the total achieved rate may be larger (smaller) than R. Rates on the subcarriers in $\{1, \ldots, N\}$ with the smallest (largest) bit increments are rounded down (up) by β bits *iteratively* till meeting $C1$. Note that in our algorithm considering M in SWF allows that only subcarriers in $\mathcal{D}$ are considered. $\lceil R_r/\beta \rceil$ subcarriers with the smallest bit increments can be rounded up at one time *non-iteratively*. Rates on other subcarriers are rounded down. By using the order statistic selection algorithms [15], this step can be efficiently implemented with complexity $\mathcal{O}(N)$ in the worst case. Instead of comparing power increments used in [8], [9], we reduce the number of exponential operations from N to $N/2$ to obtain the transmit power after rounding up and down on subcarriers in $\mathcal{D}$ by comparing rate increments, when the average of d equal to $N/2$ is assumed. To sum up, the complexity of EBL is around $\mathcal{O}(N)$.

VI. Simulation Results

In this section, simulation results are given to quantify the influence of the channel estimation error on optimal bit loading for OFDM. The frequency selective channel is modeled as consisting of 16 independent Rayleigh distributed multipaths with an exponentially decaying profile and c in (2) is set to one. The expected CNR on each subcarrier is set to 5 dB. The imperfect CSI is given by performing the least-squares channel estimation on each subcarriers. For an intended BER of 10^{-6}, $M = 6$ bits at maximum can be transmitted over each subcarrier in one OFDM symbol. The proposed algorithm is implemented with MATLAB and its computing time is measured by the pair of commands (tic, toc), which is recommended by MATLAB help.

Fig. 2 shows the transmit power of optimal bit loading with imperfect CSIs and perfect CSI over $N = 16$ subcarriers. Power increment due to imperfect CSI (around 50.6% for $I_t = 1$, around 6.5% for $I_t = 10$ and around 0.67% for $I_t = 100$) decreases as increasing I_t. However, large I_t is almost impossible in practice. On one hand, if pilot-based channel estimation is used, large number of pilot symbols would deteriorate bandwidth efficiency. On the other hand, even if blind or semi-blind channel estimation is adopted, the stability of channels within a sufficiently long period cannot be guaranteed and the unavoidable feedback delay cannot be neglected any more. For example, 24 to 36 OFDM symbols in each frame are transmitted in the worldwide interoperability

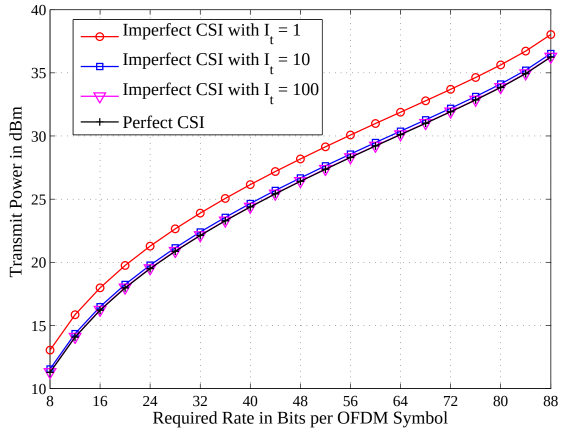

Fig. 2. Optimal bit loading for MMP with imperfect and perfect CSIs.

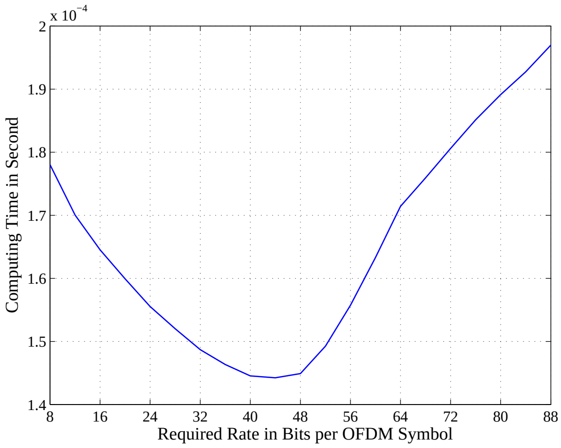

Fig. 3. Computing time for EBL vs. increasing required rate.

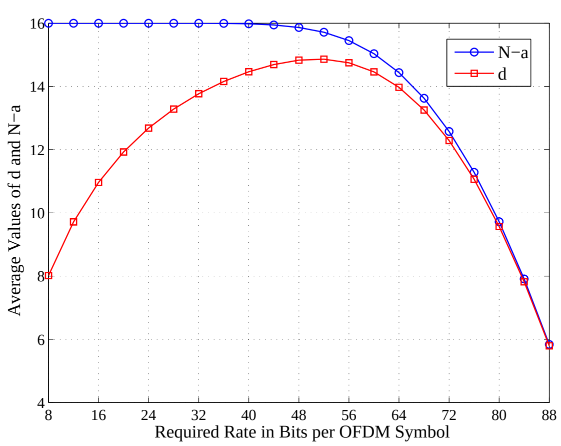

Fig. 4. Average values of d and $N - a$ vs. increasing required rate.

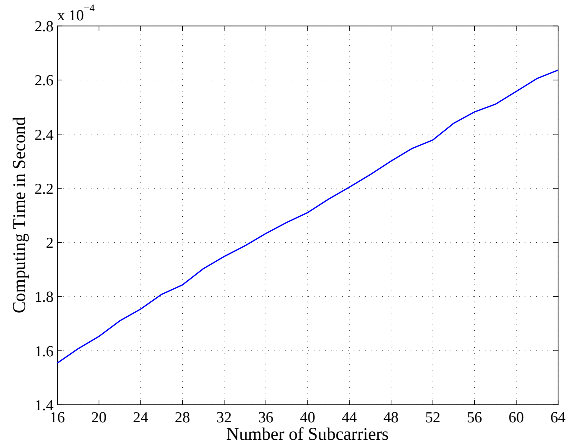

Fig. 5. Computing time for EBL vs. increasing number of subcarriers.

for microwave access (WiMAX) downlink, see [16].

The curve in Fig. 3 demonstrates the computing time for EBL against different required rates with $N = 16$. Its behavior can be explained by Fig. 4, which plots the average values of d and $N - a$ against different demanded rates. Although the gap between d and $N - a$ is very large for a small demanded rate and the sorting size shrinks to d by EBL, negative rates happen very often due to a small number of used subcarriers. For a large required rate, most of subcarriers can be used, but d approaches to $N - a$ and the constraint on the maximal allowed rate over each subcarrier needs to be solved often. A good balance is achieved for an in-between required rate. The variation coefficient of the computing time over different demanded rates is around 0.1.

Fig. 5 shows the computing time for EBL against different numbers of subcarriers N while the required rate is $R = MN/2$. It can be seen that the computing time for EBL is an approximately linearly increasing function of N. The fluctuation on this curve is induced by the variation of computing time due to different required rates. Hence, it is proved that EBL can be computed in expected $\mathcal{O}(N)$ time.

VII. Conclusion

In this paper, the performance degradation of bit loading by channel estimation error has been studied first. As an approximate approach, the CRLB has been used to express the channel estimation error. The resulting effective noise power only depends on the number of OFDM symbols aiding channel estimation in each frame. This makes the frame length be an important system performance parameter while blind or semi-blind channel estimation is adopted. This has been proved by the numerical results. Through the analysis on the continuous output rates of the modified water-filling, a non-iterative algorithm has been proposed to quantize the continuous output rates. Simulation results have shown that the computing time for optimal bit loading in OFDM systems can increase linearly with the increasing number of subcarriers.

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