Capacity and power control in spread spectrum macro-diversity radio networks revisited

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Abstract—**Macro-diversity — all base stations decode cooperatively each received signal — can mitigate shadow fading, and increase the capacity of a spread-spectrum communication network. Assuming that a terminal's transmission power contributes to its own interference, the literature determines whether a vector of quality-of-service targets is feasible through a simple formula, which is insensitive to the terminals' channel gains. Herein, through Banach' contraction-mapping principle — and without the self-interference approximation — a new low-complexity capacity formula is derived. Through its dependence on *relative* channel gains, the new formula adapts itself in a sensible manner to special conditions, such as when most terminals can only be heard by a subset of the receivers. Under such conditions, the original may significantly overestimate capacity.**

I. Introduction

Macro-diversity — when all base stations decode cooperatively each received signal — can mitigate shadow fading, and increase the capacity of a spread-spectrum communication system [1], [2], [3]. A fundamental question is whether a set of quality-of-service (QoS) targets (carrier-to-interference ratios, CIR) is feasible. This determines, for instance, whether a terminal with certain QoS requirement can be admitted, without reducing the QoS of the "incumbent" terminals. The set of all the feasible QoS vectors is associated with the "capacity region" of the system. Assuming that a transmitter's power contributes to its own interference, [2] shows that a set of QoS targets is feasible if their sum is less than the number of receivers, and that the corresponding "greedy" power adjustment process always converges to a unique power vector.

However, [2]'s formula is insensitive to the channel gains, and hence to the terminals' locations, which is somewhat counter-intuitive under certain situations. For instance, in a *K*-receiver macro-diversity scenario, if there are 2 terminals near each receiver, it must be much easier to satisfy their QoS requirements, than if the 2*K* terminals congregate near a single receiver — in which case the system behaviour should be closer to that of a one-receiver system.

Building upon [4], and without the self-interference approximation, this note derives a new macro-diversity feasibility formula whose complexity is only marginally greater than [2]'s. And through its explicit dependence on *relative* channel gains, the new formula adapts itself in a sensible manner to special conditions, such as when most terminals congregate in a small area of the system, and can therefore only be heard by a subset of the receivers.

Recent relevant work from ours includes [5], a strict generalisation of the present work, as well as [6] which — following the general approach of [7] — explores the convergence of a power adjustment process in which all the details of the system are "hidden" inside the adjusting functions.

Below, the new feasibility result is stated, interpreted and compared to the one previously available. Subsequently, after describing the basic macro-diversity system model, the derivation of the new result — which is grounded on the Banach's contraction mapping principle of fixed-point theory [8], [9] — is presented. Then, a practical admission control algorithm based on the new result is introduced, its complexity analysed, and examples of its use given and discussed. Two mathematical appendices provide the essential mathematical background, and some technical results from the literature.

II. MAIN RESULT

Consider a macro-diversity system with K receivers, and N terminals operating on the reverse link. Let $\alpha := (\alpha_1, \cdots, \alpha_N)$ be the vector of desired carrier-to-interference ratios, $h_{i,k}$ denote the channel gain in the signal from terminal i arriving at receiver k, and $g_{i,k} = h_{i,k}/h_i$ with $h_i = h_{i,1} + \cdots + h_{i,K}$. If at each receiver k and for each terminal i

$$\sum_{\substack{n=1 \\ n \neq i}}^{N} \alpha_n g_{n,k} < 1 \qquad (1)$$

then it is possible for each terminal i to operate at the CIR α_i (α belongs in the "capacity region" of the system). Furthermore, the power vector that produces α can be obtained by successive approximations, starting from an arbitrary power vector.

Condition (1) indicates that the greatest weighted sum of $N - 1$ carrier-to-interference ratios must be less than 1. The weights are relative channel gains. At most NK such simple sums need to be checked.

Condition (1) is closest to that provided by [2] in the special case in which each terminal is "equidistant" from each receiver; that is, for each i, $h_{i,k} \approx h_{i,l}$ $\forall k,l$ (for example, the terminals may be distributed along a line that is perpendicular to the axis between the 2 symmetrically placed receivers). In this case, each $g_{n,k} = 1/K$, and condition (1) reduces to $\sum_{\substack{n=1 \\ n \neq i}}^{N} \alpha_n < K$ for each i. $\sum_{\substack{n=1 \\ n \neq i}}^{N} \alpha_n$ adds all α_i except one; such sum is, evidently, largest when it leaves out the smallest α_i. By comparison, [2] gives the condition $\sum_{n=1}^{N} \alpha_n < K$ for all cases. Condition (1) is the least conservative of the two because it leaves out one α_i (the smallest) from the sum. For 3 terminals and 2 receivers, the original formula yields as achievable region the symmetric triangular pyramid with vertexes (0,0,0), (2,0,0), (0,2,0) and (0,0,2), shown in darker colour in fig. 1(a). By contrast, $\sum_{\substack{n=1 \\ n \neq i}}^{3} \alpha_n < 2$ — to which condition (1) reduces, in this example — yields the achievable region shown in fig. 1(b), which strictly contains the darker triangular pyramid of fig. 1(a), and extends to include the transparent triangular volume limited above by the line segment between (0,0,2) and (1,1,1) (indeed (0,99 , 0,99 , 0,99) does satisfy $\sum_{\substack{n=1 \\ n \neq i}}^{3} \alpha_n < 2$ but definitely *not* $\alpha_1 + \alpha_2 + \alpha_3 < 2$).

However, the original condition may also significantly overestimate system capacity, when most terminals are in effective range of only a few receivers. Suppose in the previous example that a third receiver exists, but that the terminals are located in such a way that they have a strong signal to only two receivers; that is, for each i, $h_{i,1} \approx h_{i,2}$ while $h_{i,3} \approx 0$. Thus, $g_{i,1} \approx g_{i,2} \approx 1/2$ and $g_{i,3} \approx 0$. Then, condition (1) still reduces to $\sum_{\substack{n=1 \\ n \neq i}}^{N} \alpha_n < 2$ for each i, and leads to the already discussed achievable region. However, the original condition leads to $\sum_{n=1}^{N} \alpha_n < 3$, which, as illustrated by figure 1(c), greatly overestimates (for the special case considered) the achievable region, by extending it to the triangular planar region with vertexes (3,0,0),(0,3,0) and (0,0,3)) (see also subsection VII-C2).

Let us now consider the simple *asymmetric* case of 3 terminals and 2 receivers, with relative gains to the first receiver of 2/3, 1/3, and 1/2, respectively. Condition (1) leads to 3 inequalities per receiver, such as $\frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 < 1$, $\frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2 < 1$, $\frac{2}{3}\alpha_2 + \frac{1}{2}\alpha_3 < 1$, etc. The combination of these inequalities yields a region illustrated by figures 2(a) and 2(b), which is limited from above by the line segment between (0,0,2) and (1,1,2/3). As already discussed, the result from [2], $\sum_{n=1}^{N} \alpha_n < 2$, yields a symmetric pyramidal region with vertexes at (2,0,0), (0,2,0) and (0,0,2) (recall fig. 1(a)) which, as illustrated by fig. 2(c), intersects with — but neither contains nor is contained by — the region described by fig. 2(b).

As discussed further in section VII, condition (1) yields a low-complexity algorithm for admission-control decisions, which adapts itself in a sensible manner to special situations in which most terminals are in range of only a few receivers. The original condition cannot adapt to such situations because it is independent of the channel gains, and accordingly may yield over-optimistic results. It is important to notice that the
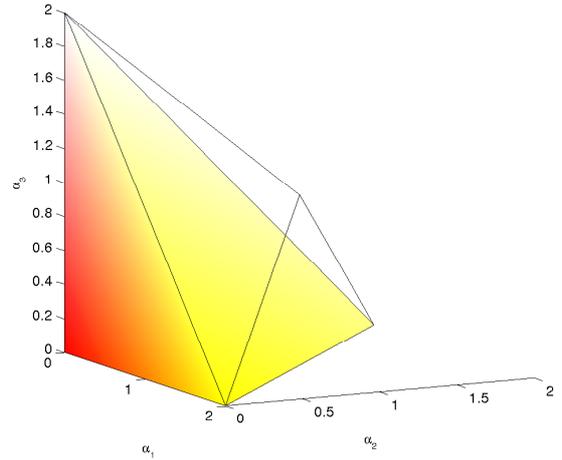
(a) The original gives as achievable region the darker pyramidal region.
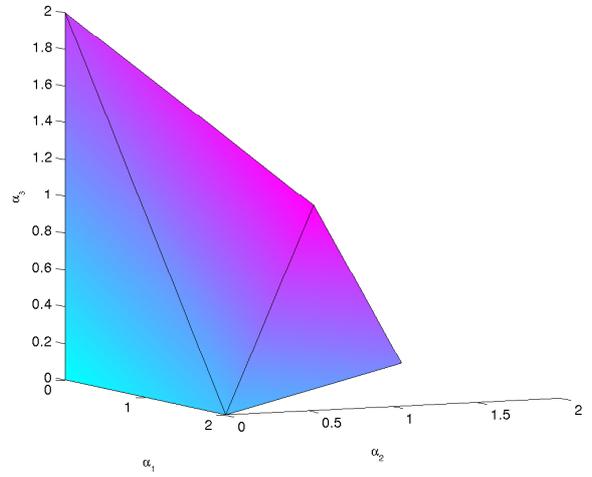
(b) The new achievable region
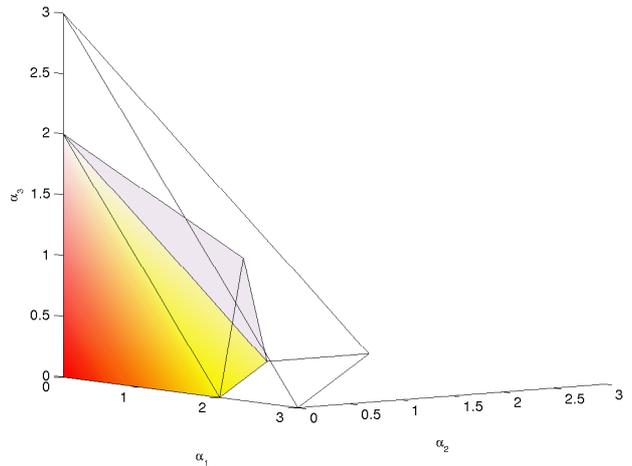
(c) When only 2 of 3 receivers can "hear" the terminals, the original greatly overestimates the achievable region

Figure 1. The new condition versus the original: Achievable regions when each of 3 terminals is "equidistant" from each receiver.
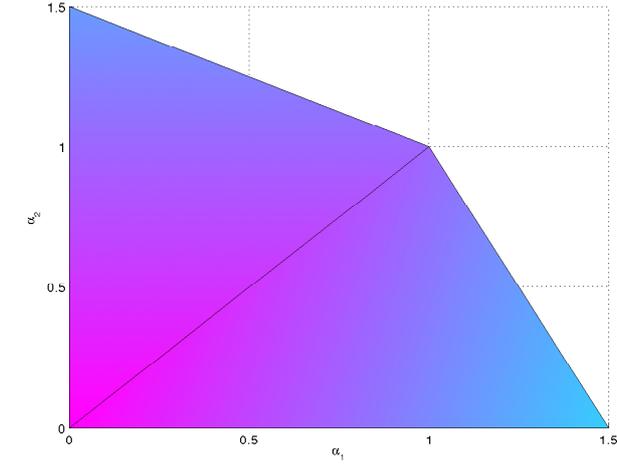
(a) View from top
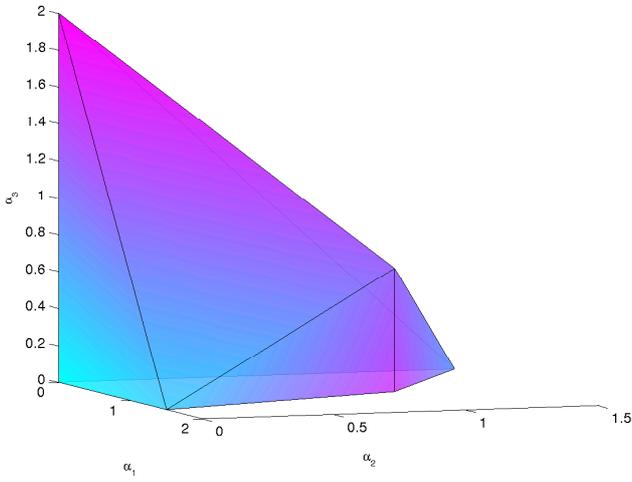
(b) Perspective view
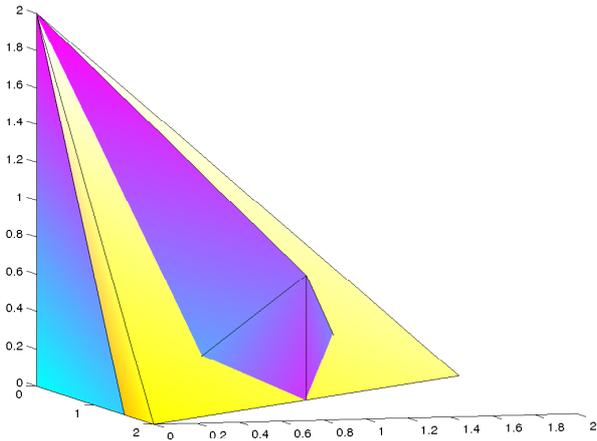
(c) The original formula produces an achievable region that neither includes nor is included by the new region

Figure 2. The macro-diversity achievable region for 3 terminals and 2 receivers, for specific asymmetric channel gains

original seems to misbehave under specific scenarios where certain key assumptions underlying [2] may not be satisfied (e.g., the "non-overcrowding" condition, and that all receivers can "hear" all transmissions). Indeed, [1, section 9.8] studies a situation where, depending on each transmitter's location, it is only "heard" by 2 of the 3 available receivers, and concludes that the simple formula of [2] should be replaced by a somewhat more elaborate condition. Nevertheless, the fact that condition (1) "automatically" adjusts itself to handle special scenarios such as that of [1, section 9.8] seems to be a major advantage.

III. THE MACRO-DIVERSITY SYSTEM MODEL

Under macro-diversity, the cellular structure is removed and each transmitter is jointly decoded by all receivers[2]. A relevant quality-of-service (QoS) index for terminal i is the product of its spreading gain by its "carrier to interference ratio" (CIR), α_i, defined as [1], [2] :

$$\alpha_i = \frac{P_i h_{i,1}}{Y_{i,1} + \sigma_1^2} + \cdots + \frac{P_i h_{i,K}}{Y_{i,K} + \sigma_K^2} \qquad (2)$$

where P_i is the transmission power level of terminal i, K is the number of receivers in the network, $h_{i,k}$ is the channel gain in the signal from terminal i arriving at receiver k, σ_k^2 is the (average) power of the additive random noise at receiver k, and $Y_{i,k}$ denotes the interfering power experienced by transmitter i at receiver k; i.e.,

$$Y_{i,k} := \sum_{\substack{n=1 \\ n \neq i}}^{N} P_n h_{n,k} \qquad (3)$$

Below, we utilise the interference vector

$$\mathbf{Y}_i := (Y_{i,1}, \cdots, Y_{i,K}) \qquad (4)$$

It will prove useful to define the *relative* channel gains:

$$g_{i,k} := \frac{h_{i,k}}{h_i} \qquad (5)$$

where,

$$h_i := h_{i,1} + \cdots + h_{i,K} \qquad (6)$$

IV. MACRO-DIVERSITY CAPACITY ISSUES

A. The Capacity question

Conditions are sought under which a given N-vector of positive numbers, $\alpha := (\alpha_1, \cdots, \alpha_N)$, is such that there exists another N-vector of positive numbers, $\mathbf{P} = (P_1, \cdots, P_N)$, satisfying appropriate constraints, and equation (2) for each i; that is, the system formed by N equations of the form (2) has a solution in the appropriate space. When such solution exists, the vector of power ratios α is said to be in the "capacity region" of the system.

More formally, let $\mathcal{P}$ denote the set of admissible power vectors, and $\alpha_i(\mathbf{P})$ be defined by (2). The capacity region of the system is then defined as

$$\mathcal{C} = \{(\alpha_1(\mathbf{P}), \cdots, \alpha_N(\mathbf{P})) | \mathbf{P} \in \mathcal{P}\} \qquad (7)$$

Relevant discussions of the capacity region of power-controlled cellular systems — albeit without macro-diversity — can be found in [10], [11], [12].

B. Macro-diversity capacity as a fixed-point problem

One can envision a process in which terminals take turns adjusting transmission power, and each terminal chooses the power level that achieves its desired CIR under the present level of interference. Of course, when a terminal changes its power, it also alters the level of interference experienced by the others, which may lead to further power adjustments by other terminals.

For instance, from equation (2) one obtains the power adjustment process

$$P_i = \alpha_i \left(\frac{h_{i,1}}{Y_{i,1} + \sigma_1^2} + \cdots + \frac{h_{i,K}}{Y_{i,K} + \sigma_K^2} \right)^{-1} \quad (8)$$

If this adjustment process "converges" to a feasible power vector, in the sense that at the present power levels, no terminal needs further power adjustment to achieve its desired CIR, evidently, those CIR's are feasible.

If we denote as $\mathbf{T}(\mathbf{p})$ the function that produces a new power vector as a function of the present one, when each terminal adjusts its power as described above, then we need to explore the conditions under which there is a vector $\mathbf{p}^*$ such that $\mathbf{p}^* = \mathbf{T}(\mathbf{p}^*)$, that is, $\mathbf{p}^*$ is a fixed point of $\mathbf{T}$.

C. Banach' fixed-point principle

For some transformation $\mathbf{T}$ from certain space into itself, fixed-point theory explores conditions under which $\mathbf{T}$ has a fixed-point, that is, there is a point $\mathbf{x}^*$ in the concerned space such that $\mathbf{x}^* = \mathbf{T}(\mathbf{x}^*)$. In particular, Theorem B.1 holds that, if $\mathbf{T}$ is a *contraction* (Definition B.1), then $\mathbf{T}$ has a *unique* fixed-point, and that it can be found iteratively via successive approximation (Definition B.2).

D. Normalised adjustment

Equation (8) involves a ratio of sums of ratios with which is very difficult to work. Reference [2] simplifies the macro-diversity analysis by including a terminal's own signal as part of the interference (thus, the sum in equation (3) is taken over *all n*). As an alternative, in equation (8), one can replace each $Y_{i,k}(\mathbf{P})$ with

$$\hat{Y}_i := \max_k \{Y_{i,k}\} \quad (9)$$

and each σ_k^2 with

$$\hat{\sigma} := \max_k \{\sigma_k^2\} \quad (10)$$

Then, equation (8) becomes (recall that $h_i = \sum_k h_{i,k}$, eq. (6)):

$$P_i = \frac{\alpha_i}{h_i} \left(\hat{Y}_i + \hat{\sigma} \right) \quad (11)$$

Equation (11) strictly overestimates the amount of power that terminal i needs in order to achieve its desired CIR. Thus, it stands to reason that if a set of CIR is feasible under the adjustment process given by Equation (11), it is also feasible under the original adjustment.

E. Properties of the new macro-diversity adjustment

Notice that $\hat{Y}_i \equiv \|\mathbf{Y}_i\|_\infty$ (Definition A.4). Thus, the adjustment process can now be written as $P_i = f_i(\mathbf{P}_{-i}) + c_i$ where,

$$f_i(\mathbf{P}_{-i}) := \frac{\alpha_i}{h_i} \|\mathbf{Y}_i(\mathbf{P}_{-i})\|_\infty \quad (12)$$

and

$$c_i := \frac{\alpha_i}{h_i} \hat{\sigma} \quad (13)$$

Equation (12) involves a "norm" (Definition A.2). This suggests that f_i may satisfy Definition A.1. Clearly, if $g_i(\mathbf{P}_{-i}) := \|\mathbf{Y}_i(\mathbf{P}_{-i})\|$ satisfies Definition A.1, so does f_i. We show below that g_i does satisfy Definition A.1.

Lemma IV.1: For $\mathbf{x} \in \Re^M$ and $k = 1, \cdots, K$, consider the vectors $\mathbf{a}_k = (a_{1,k}, \cdots, a_{M,k})$ with $a_{m,k} > 0$, and let $v_k(\mathbf{x}) = \sum_{m=1}^M a_{m,k} |x_m|$, $\mathbf{v}(\mathbf{x}) := (v_1(\mathbf{x}), \cdots, v_K(\mathbf{x}))$, and $f(\mathbf{x}) := \|\mathbf{v}(\mathbf{x})\|_\beta$, where $\|\cdot\|_\beta$ denotes a norm on $\Re^K$ (Definition A.2). Then f satisfies Definition A.1.

Proof: It is evident that f satisfies properties 1 and 2 of Definition A.1. To check property 3, consider $f(\mathbf{x} + \mathbf{y}) = \|\mathbf{v}(\mathbf{x} + \mathbf{y})\|_\beta$

$$\begin{aligned} v_k(\mathbf{x} + \mathbf{y}) &= \sum_{m=1}^M a_{m,k} |x_m + y_m| \leq \sum_{m=1}^M a_{m,k}(|x_m| + |y_m|) \\ &\equiv \sum_{m=1}^M a_{m,k} |x_m| + \sum_{m=1}^M a_{m,k} |y_m| \\ &\equiv v_k(\mathbf{x}) + v_k(\mathbf{y}) \end{aligned}$$

Thus,

$$\begin{aligned} f(\mathbf{x} + \mathbf{y}) &\leq \| \begin{bmatrix} v_1(\mathbf{x}) + v_1(\mathbf{y}) & \cdots & v_K(\mathbf{x}) + v_K(\mathbf{y}) \end{bmatrix}^T \|_\beta \\ &\equiv \|\mathbf{v}(\mathbf{x}) + \mathbf{v}(\mathbf{y})\|_\beta \end{aligned}$$

By hypothesis, $\|\cdot\|_\beta$ has property 3. Therefore, $\|\mathbf{v}(\mathbf{x}) + \mathbf{v}(\mathbf{y})\|_\beta \leq \|\mathbf{v}(\mathbf{x})\|_\beta + \|\mathbf{v}(\mathbf{y})\|_\beta \equiv f(\mathbf{x}) + f(\mathbf{y})$.

Thus, $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$ and f satisfies Definition A.1. ∎

Remark 1: It is evident and consistent with Theorem A.1 that the function f is monotonic (Definition A.7).

V. THE BANACH APPROACH APPLIED TO MACRO-DIVERSITY

Below we address the macro-diversity capacity question through fixed-point theory, specifically through Theorem B.1, the Banach Contraction Mapping Principle.

Let the transformation $\mathbf{T}(\mathbf{P})$ be defined by

$$\begin{bmatrix} T_1(\mathbf{P}) \\ \vdots \\ T_N(\mathbf{P}) \end{bmatrix} = \begin{bmatrix} \breve{f}_1(\mathbf{P}) + c_1 \\ \vdots \\ \breve{f}_N(\mathbf{P}) + c_N \end{bmatrix} \quad (14)$$

with

$$\breve{f}_i(\mathbf{x}) := 0 \cdot x_i + f_i(\mathbf{x}_{-i}) \equiv f_i(\mathbf{x}_{-i}) \quad (15)$$

for $x \in \Re^N$, where f_i and c_i are given by eqs. (12) and (13), respectively.

Below we characterise the conditions under which $\mathbf{T}$ is a contraction, which, by Theorem B.1, implies that $\mathbf{T}$ has a *unique* fixed-point, and that it can be found by successive approximations (Definition B.2).

Remark 2: One can choose any norm to apply Theorem B.1. Below we utilise the infinity norm (Definition A.4). The sub-index of $\|\cdot\|_\infty$ is omitted for notational convenience.

Theorem V.1: Let $\vec{1}_M$ denote the element of $\Re^M$ with each component equal to 1. If each f_i satisfies Definition A.1, and is such that $\forall i, f_i(\vec{1}_{N-1}) < 1$ then the transformation $\mathbf{T}$ defined by eq. (14) satisfies Definition B.1.

Proof: By Lemma IV.1, each f_i satisfies Definition A.1. Fixed-point analysis necessitates functions defined on $\Re^N$, but it is a simple matter to show that if f_i satisfies Definition A.1 on $\Re^{N-1}$, then $\check{f}_i$ satisfies Definition A.1 as a function on $\Re^N$.

For $\mathbf{x}, \mathbf{y} \in \Re^N$, $\|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})\| =$

$$\left\| \begin{bmatrix} \check{f}_1(\mathbf{x}) - \check{f}_1(\mathbf{y}) \\ \vdots \\ \check{f}_N(\mathbf{x}) - \check{f}_N(\mathbf{y}) \end{bmatrix} \right\| \equiv \max \begin{bmatrix} |\check{f}_1(\mathbf{x}) - \check{f}_1(\mathbf{y})| \\ \vdots \\ |\check{f}_N(\mathbf{x}) - \check{f}_N(\mathbf{y})| \end{bmatrix} \tag{16}$$

By Lemma A.1, $|\check{f}_i(\mathbf{x}) - \check{f}_i(\mathbf{y})| \leq \check{f}_i(\mathbf{x} - \mathbf{y})$. Thus,

$$\max \begin{bmatrix} |\check{f}_1(\mathbf{x}) - \check{f}_1(\mathbf{y})| \\ \vdots \\ |\check{f}_N(\mathbf{x}) - \check{f}_N(\mathbf{y})| \end{bmatrix} \leq \max \begin{bmatrix} \check{f}_1(\mathbf{x} - \mathbf{y}) \\ \vdots \\ \check{f}_N(\mathbf{x} - \mathbf{y}) \end{bmatrix} \tag{17}$$

Let $M_{x,y} := \max(|x_1 - y_1|, \cdots, |x_N - y_N|) \equiv \|\mathbf{x} - \mathbf{y}\|$

By monotonicity (see Remark 1),

$$\check{f}_i(\mathbf{x} - \mathbf{y}) \leq \check{f}_i(M_{xy}, \cdots, M_{xy}) \equiv \check{f}_i(M_{xy}\vec{1}_N) \tag{18}$$

By homogeneity (condition (2) of Definition A.1)

$$\check{f}_i(M_{xy}\vec{1}_N) = M_{xy}\check{f}_i(\vec{1}_N) \equiv \|\mathbf{x} - \mathbf{y}\|\check{f}_i(\vec{1}_N) \equiv \|\mathbf{x} - \mathbf{y}\|f_i(\vec{1}_{N-1}) \tag{19}$$

Thus,

$$\max \begin{bmatrix} |\check{f}_1(\mathbf{x}) - \check{f}_1(\mathbf{y})| \\ \vdots \\ |\check{f}_N(\mathbf{x}) - \check{f}_N(\mathbf{y})| \end{bmatrix} \leq \lambda \|\mathbf{x} - \mathbf{y}\| \tag{20}$$

with $\lambda := \max\{f_1(\vec{1}_{N-1}), \cdots, f_N(\vec{1}_{N-1})\} < 1$.
Therefore, $\|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})\| \leq \lambda \|\mathbf{x} - \mathbf{y}\|$ with $\lambda \in [0, 1)$. ∎

VI. CAPACITY IMPLICATIONS

A. Original coordinates

The feasibility condition of Theorem V.1 when applied to the adjustment rule of section IV-D leads to:

$$\alpha_i \sum_{\substack{n=1 \\ n \neq i}}^N \frac{h_{n,k}}{h_i} < 1 \quad \forall i, k \tag{21}$$

B. New coordinates

Condition (21) can be improved upon through a change of coordinates. Equation (11) suggests the change of variable:

$$q_i := \frac{h_i P_i}{\alpha_i} \tag{22}$$

Now, $P_n h_{n,k} \equiv q_n \alpha_n h_{n,k}/h_n \equiv q_n \alpha_n g_{n,k}$ (recall that $g_{i,k} := h_{i,k}/h_i$ with $h_i = \sum_k h_{i,k}$). Corresponding to equation (3), we now have

$$Y_{i,k} := \sum_{\substack{n=1 \\ n \neq i}}^N q_n \alpha_n g_{n,k} \tag{23}$$

The adjustment process given by equation (11) can be expressed under the new coordinates, as $q_i = g_i(\mathbf{q}_{-i}) + \hat{\sigma}$ with

$$g_i(\mathbf{q}_{-i}) := \max_k \sum_{\substack{n=1 \\ n \neq i}}^N q_n \alpha_n g_{n,k} \equiv \|\mathbf{Y}_i(\mathbf{q}_{-i})\|_\infty \tag{24}$$

Now, the feasibility condition leads to

$$\max_{i,k} \sum_{\substack{n=1 \\ n \neq i}}^N \alpha_n g_{n,k} < 1 \tag{25}$$

VII. MACRO-DIVERSITY ADMISSION CONTROL

Condition (25) can be very useful in making admission control decisions in a macro-diversity system. The basic admission-control problem can be stated as follows: Given N "incumbent" terminals with known channel states, and each achieving its desired carrier-to-interference ratio (CIR) α_i, can another terminal with known channel state that wishes a given CIR α_{N+1} be admitted into the system and provide to each terminal its desired quality-of-service level? The answer is affirmative if condition (25) is satisfied with the parameters of the incumbent and the entering terminals.

A. The feasibility condition re-formulated

For admission decisions it may be preferable to re-write condition (25) as follows.
Let

$$S_k(M) := \sum_{m=1}^M \alpha_m g_{m,k} \tag{26}$$

Thus, condition (25) can be re-stated as

$$S_k(N) - \alpha_i g_{i,k} < 1 \quad \forall i, k \tag{27}$$

The left side of inequality (27) is largest when the term being subtracted is smallest. Thus, with j_k defined as

$$\alpha_{j_k} g_{j_k,k} \leq \alpha_n g_{n,k} \quad \forall n \tag{28}$$

that is, j_k is the index of the terminal that has the smallest $\alpha_i g_{i,k}$ product, condition (27) can be re-stated as follows.
At each receiver k,

$$S_k(N) - \alpha_{j_k} g_{j_k,k} < 1 \tag{29}$$

Thus, once the terminal with the smallest $\alpha_i g_{i,k}$ product is identified at each receiver, only one inequality, (29), need to be checked per receiver.

B. The admission control algorithm

The admission problem can now be stated with the notation of section VII-A. At a given admission point, there are N terminals operating with known channel gains. $S_k(N)$ and j_k are known, and condition (29) is satisfied at each receiver. Terminal $N+1$, whose channel gain to each receiver is known, wishes to join the incumbent terminals and obtain a CIR α_{N+1}. Can this terminal wishes be granted, without reducing the QoS of any incumbent below its desired level?

From the development of section VII-A, it is clear that in order to answer the admission question, the following steps should be performed:

1) Calculate the K products $\beta_k := \alpha_{N+1} g_{N+1,k}$
2) For each receiver, obtain $S_k(N+1) = S_k(N) + \beta_k$
3) Update j_k if necessary; that is, if for some k, $\beta_k < \alpha_{j_k} g_{j_k,k}$ then $N+1 \rightarrow j_k$.
4) If at each k, $S_k(N+1) - \alpha_{j_k} g_{j_k,k} < 1$, then admit the terminal wishing service.

Notice that the 4 steps above involve very simple operations. K multiplications, K additions, K comparisons of two real numbers, K simple subtractions, and K comparisons between a number and one. These simple operations are to be performed by the system controller, which presumably has at its disposal plenty of computational resources, and no significant energy constraint. Thus, complexity does not appear to be a problem.

In fact, the algorithm may be optimised further. For instance, if after the second step, it turns out that $S_k(N+1) < 1$ for a subset of the receivers, steps 3 and 4 need not be performed for those receivers. And if $S_k(N+1) < 1$ for each k, then the terminal can be admitted without further calculations because condition (29) will be necessarily satisfied with the expanded set of terminals.

In practise, since channel gains do change over time, the system will need to periodically update all the parameters used in feasibility and admission calculations. Also, when a terminal exits the system, $S_k(\cdot)$ and possibly j_k need updating.

C. Some special cases

One can gain further insight into the admission control condition by considering certain special cases.

1) All incumbents near a given receiver: Suppose each incumbent is very close to receiver one, and relatively far to all other receivers. Then, for each i, $g_{i,1} \approx 1$ and, for $k > 1$, $g_{i,k} \approx 0$. Strictly for expository convenience, suppose further that $\alpha_N \leq \alpha_i \ \forall i$. Clearly, $S_k(N) \approx 0$ for $k > 1$; thus, for $k > 1$ the left side of condition (29) is zero, and there is nothing further to check. For $k = 1$, condition (29) becomes

$$S_1(N) - \alpha_N = \sum_{n=1}^{N-1} \alpha_n < 1 \tag{30}$$

Suppose that another terminal wants to enter the system, and achieve CIR α_{N+1}. Intuitively, the location of this terminal should make a big difference in the admission decision. If it wants to enter the system in the "crowded" region admission should be more difficult than if it wants to enter a distant, less congested area. Let us see how the algorithm of section VII-B handles this situation.

a) Entrance into the crowded area: If the new terminal is in the same area as the others, then its relative channel gains satisfy $g_{N+1,1} \approx 1$ and, for $k > 1$, $g_{N+1,k} \approx 0$. It follows that $\alpha_{N+1} g_{N+1,k} \approx 0$ for $k > 1$. Thus the feasibility condition is in doubt only for $k = 1$. Now, $S_1(N+1) = \sum_{n=1}^{N+1} \alpha_n$, and if α_N continues to be the smallest CIR, then the terminal can be admitted only if $\sum_{n=1}^{N-1} \alpha_n + \alpha_{N+1} < 1$.

b) Entrance near a distant receiver: If the new terminal is near a distant receiver, say K, it means that $g_{N+1,K} \approx 1$ and $g_{N+1,k} \approx 0$ for $k < K$. Because $g_{N+1,1} \approx 0$, admitting the new terminal has no effect on the situation near receiver 1: $S_1(N+1) = S_1(N)$, and the feasibility inequality (30) will continue to hold. Near receiver K the new terminal faces no interference, thus the left side of condition (29) is zero. Thus, condition (29) would be satisfied at each k if the new terminal joins the system; thus, it can be admitted.

c) Conclusion for this case: The algorithm of section VII-B behaves sensibly. It will definitely admit a new terminal that is located near a receiver that is distant from the crowded area. But if the new terminal wants to join the crowd, the admission decision depends on the specific parameters of the entering and incumbent terminals.

2) All terminals between 2 receivers: Suppose now that all incumbents and the entering terminal are located between receivers 1 and 2, so that, for each i, $g_{i,1} \approx g_{i,2} \approx 1/2$ and $g_{i,k} \approx 0$ for $k > 2$. In this case, the left side of condition (29) is zero for $k > 2$. The feasibility condition is only in doubt, concerning receivers 1 and 2. Assuming, for notational convenience, that $\alpha_{N+1} \leq \alpha_i \ \forall i$, the feasibility condition for $k \in \{1,2\}$ with $N+1$ terminals reduces to $S_k(N+1) - \frac{1}{2}\alpha_{N+1} = \frac{1}{2}\sum_{n=1}^{N} \alpha_n < 1$ or

$$\sum_{n=1}^{N} \alpha_n < 2 \tag{31}$$

The condition $\sum_{n=1}^{N+1} \alpha_n < K$ — given by [2] for all cases independently from the channel gains — would have slightly increased the left side of (31) by adding all CIR, but if $K \gg 2$, it may have greatly over-estimated the right side of (31), and hence the admission capability of the system, for this case.

Channel gains also play a prominent role in the feasibility analysis of other multi-cell CDMA systems, such as in [13].

APPENDIX A
NORMS AND RELATED MATERIAL

A. Concepts and definitions

Let V denote a vector space (for a formal definition of these spaces see [14, pp. 11-12]).

Definition A.1: A function $f: V \rightarrow \Re$ is called a *semi-norm* on V, if it satisfies:

1) $f(v) \geq 0$ for all $v \in V$ (non-negativity)
2) $f(\lambda v) = |\lambda| \cdot f(v)$ for all $v \in V$ and all $\lambda \in \Re$ (homogeneity)
3) $f(v+w) \leq f(v) + f(w)$ for all $v, w \in V$ (sub-additivity or "the *triangle inequality*")

Definition A.2: If a semi-norm additionally satisfies $f(v) = 0$ if and only if $v = \theta$ (where θ denotes the zero element of V), then f is called a norm on V and $f(v)$ is usually denoted as $\|v\|$.

Remark A.1: It is a simple matter to show that a function that satisfies properties 2 and 3 above is convex. Thus, (semi-)norm-minimisation problems are often well-behaved.

Definition A.3: The Hölder norm with parameter $p \geq 1$ ("p-norm") is denoted as $\|\cdot\|_p$ and defined for $x \in \Re^N$ as

$$\|\mathbf{x}\|_p = (|x_1|^p + \cdots + |x_N|^p)^{\frac{1}{p}} \qquad \text{(A.1)}$$

Remark A.2: With $p = 2$, the Hölder norm becomes the familiar Euclidean norm. The $p = 1$ case is also often encountered. Furthermore, it can be shown that $\lim_{p \to \infty} \|\mathbf{x}\|_p = \max(|x_1|, \cdots, |x_N|)$, which leads to the following definition:

Definition A.4: For $x \in \Re^N$, the supremum or infinity norm is denoted as $\|\cdot\|_\infty$ and defined as

$$\|\mathbf{x}\|_\infty := \max(|x_1|, \cdots, |x_N|) \qquad \text{(A.2)}$$

Definition A.5: For $\mathbf{v} \in \Re^N$, let $\mathbf{w}$ be such that $w_i := |v_i|$, and denote $\mathbf{w}$ as $|\mathbf{v}|$.

Definition A.6: A norm, $\|\cdot\|$, on $\Re^N$ is called an *absolute vector norm* if it depends only on the absolute values of the components of the vector; that is, for $\mathbf{v} \in \Re^N$, and $\mathbf{w}$ such that $w_i := |v_i|$, $\|\mathbf{v}\| \equiv \|\mathbf{w}\|$.

Definition A.7: For $\mathbf{x}$ and $\mathbf{y} \in \Re^N$, let $\mathbf{x} \leq \mathbf{y}$ mean that $x_i \leq y_i$ for each i. A norm, $\|\cdot\|$, on $\Re^N$ is said to be *monotonic* if, for any $\mathbf{x}$ and $\mathbf{y} \in \Re^N$, $|\mathbf{x}| \leq |\mathbf{y}|$ implies that $\|\mathbf{x}\| \leq \|\mathbf{y}\|$.

B. Useful results from the literature

Lemma A.1: (Reverse triangle inequality) If the function $f : V \to \Re$ satisfies the triangle inequality, then $|f(\mathbf{x}) - f(\mathbf{y})| \leq f(\mathbf{x} - \mathbf{y})$.

Proof: Without loss of generality, suppose that $f(\mathbf{x}) \geq f(\mathbf{y})$ which implies that $f(\mathbf{x}) - f(\mathbf{y}) \equiv |f(\mathbf{x}) - f(\mathbf{y})|$.

Observe that $\mathbf{x} \equiv (\mathbf{x} - \mathbf{y}) + \mathbf{y}$ and apply the triangle inequality to this sum:

Thus, $f(\mathbf{x}) \equiv f((\mathbf{x} - \mathbf{y}) + \mathbf{y}) \leq f(\mathbf{x} - \mathbf{y}) + f(\mathbf{y})$ or

$$f(\mathbf{x}) - f(\mathbf{y}) = |f(\mathbf{x}) - f(\mathbf{y})| \leq f(\mathbf{x} - \mathbf{y}) \qquad \text{(A.3)}$$

∎

Remark A.3: Through (A.3) one can prove that all norms are continuous.

Theorem A.1: A norm on $\Re^N$ is monotonic if and only if it is an absolute vector norm.

Proof: See [15]. ∎

APPENDIX B
BANACH FIXED-POINT THEORY

Below, the version of the Banach contraction mapping theorem for a *normed* space (as [8, Theroem 6]) is given. Often the somewhat more general form applicable on a *metric* space is presented (e.g., [9, Theorem 3.1.2, p. 74]).

Definition B.1: A map T from a normed space $(V, \|\cdot\|)$ into itself is a *contraction* if there exists $\lambda \in [0, 1)$ such that for all x, $y \in V$, $\|T(x) - T(y)\| \leq \lambda \|x - y\|$.

Definition B.2: (Successive approximation) For expository convenience, let $T^m(x_1)$ for $x_1 \in V$ be defined inductively by $T^0(x_1) = x_1$ and $T^{m+1}(x_1) = T(T^m(x_1))$, with $m \in \{1, 2, \cdots\}$.

Theorem B.1: (Banach' Contraction Mapping Principle) If T is a contraction mapping on V there is a unique $x^* \in V$ such that $x^* = T(x^*)$. Moreover, x^* can be obtained by successive approximation, starting from an arbitrary initial $x_0 \in V$; i.e., for any $x_0 \in V$, $\lim_{m \to \infty} T^m(x_0) = x^*$.

Proof: See [8][9, Theorem 3.1.2, p. 74]. ∎

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