Symmetric Degrees of Freedom of the MIMO 3-Way Channel with $M_{\mathrm{T}} \times M_{\mathrm{R}}$ Antennas

Henning Maier*, Anas Chaaban†, and Rudolf Mathar*
*Institute for Theoretical Information Technology, RWTH Aachen University, 52056 Aachen, Germany
†Institute of Digital Communication Systems, Ruhr-Universität Bochum, 44780 Bochum, Germany
Email: {maier, mathar}@ti.rwth-aachen.de, {anas.chaaban}@rub.de

***Abstract*—We investigate the symmetric Degrees-of-Freedom (DoF) of a homogeneous multiple-input multiple-output (MIMO) 3-way channel with M_{T} transmit antennas and M_{R} receive antennas at each user. The 3-way channel extends the two-way channel to three users, who exchange six messages in total, i. e., there is one message from each user to each of the two other users. We assume that each user operates in a perfect full-duplex mode. Genie-aided upper bounds on the DoF of the channel are derived and it is shown that those are achievable by combining MIMO interference alignment, null-space beam-forming and zero-forcing. A particular gain from this homogeneous setup is that the symmetric DoF allocation providing complete fairness among all users is sum-DoF optimal.**

I. Introduction

A natural communication scenario with multiple users is multi-way device-to-device (D2D) conferencing. Especially in wireless multi-user communication networks, this will involve multiple simultaneous transmissions causing interference that impairs the maximal achievable data rates per user. The D2D approach [1]–[3] intends to increase the spectral efficiency of direct link multi-way networks without the utilization of base stations for data transmission (except for low-rate top-level control mechanisms). As a countermeasure to deal with the impairment caused by interference, all transmission signals must be carefully designed so that interference is minimized. For multi-way conferencing situations with eminently high and almost symmetric rate demands, as in video conferences for instance, using devices homogeneously equipped with the same number of antennas is beneficial. Moreover, the homogeneity provides a more convenient approach to derive efficient communication schemes and it is evident that such communication scenarios provide higher symmetric rate gains than heterogeneous scenarios.

We employ multiple-input multiple output (MIMO) interference alignment (IA) as introduced in [4] and [5] providing a key method to efficiently achieve high data rates in the presence of multi-user interference. Many works dealing with MIMO IA mainly investigate the degrees-of-freedom (DoF) [6] as a capacity approximation which becomes accurate in the high signal-to-noise ratio (SNR) regime. The DoF of several unidirectional multi-user interference networks have already

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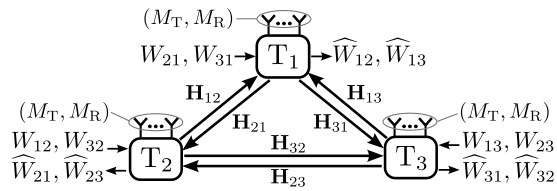

Fig. 1. The homogeneous MIMO 3-way channel (or Δ-channel) with M_{T} transmit and M_{R} receive antennas at each user T_i, with $i = 1, 2, 3$.

been studied thoroughly. A particular focus is set on the DoF for MIMO IA with constant channel coefficients. For instance, the DoF of the 2-user MIMO interference channel using zero-forcing are provided in [6], the DoF and the DoF region of the 2-user MIMO X-channel are considered in [5], [7], respectively, where MIMO IA was used. In several studies, the devices are assumed to have homogeneous antenna configurations. For instance the DoF of a (homogeneous) 3-user MIMO interference channel with M antennas per transmitter and N antennas per receiver are derived in [8], [9]. The DoF of the general MIMO K-user interference channel with a heterogeneous number of antennas at the transmitters and receivers is yet unknown and quite challenging to derive so far.

In this work, we investigate a MIMO 3-way channel as depicted in Figure 1. It can be considered as an extension of Shannon's two-way channel [10] to three users with multiple antennas. Three full-duplex users intend to exchange messages with each other directly. A related 3-user multi-way channel has been considered earlier in the context of multi-way relay channels: The DoF of the MIMO 3-way relay channel, also known as the Y-channel, is studied in [11] and [12]. The Y-channel is a relay-aided counterpart to the 3-way channel. We consider multi-way communications without a dedicated relay node contrary to [11] and [12]. The single-input single-output (SISO) variant of the 3-way channel is studied in [13], where the sum-capacity is characterized within 2 bits. The result of [13] states that the sum-capacity can be approached by letting the two strongest users communicate while leaving the third one silent - this is clearly not a fair scheme. In the present $M_{\mathrm{T}} \times M_{\mathrm{R}}$ setup for M_{T} transmit antennas and M_{R} receive antennas, each subchannel is equally strong (i. e., with the same rank). We derive cut-set bounds and genie-aided upper bounds to obtain a sum-DoF bound of the channel. We propose a MIMO IA and zero-forcing scheme to show that the derived sum-DoF upper bound is achievable. Moreover, the achievable schemes use symmetric DoF allocations providing completely fair rates among each user. We also observe an interesting symmetry within the parameter plane of the sum-DoF in terms of M_{T} and M_{R}.

Organization. The system model of the MIMO 3-way channel is provided in Section II. In Section III, cut-set and genie-aided upper bounds on the DoF are derived. The transmission schemes based on IA, null-space beamforming and zero-forcing are described in Section IV, achieving the sum-DoF of the channel for a symmetric DoF allocation per user. We briefly discuss the mentioned symmetry of the sum-DoF parameter plane for M_{T} and M_{R} in Section V.

Notation. We denote matrices by boldface upper case letters, e. g., $\boldsymbol{A}$, and vectors by boldface lower case letters, e. g., $\boldsymbol{a}$. The length-N sequence $(\boldsymbol{a}(1),\cdots,\boldsymbol{a}(N))$ is denoted by $\boldsymbol{a}^N$. $\boldsymbol{A}^{\mathsf{T}}$ denotes the transposed matrix of $\boldsymbol{A}$, $\boldsymbol{A}^{\dagger}$ its left Moore-Penrose pseudo-inverse, and $\mathrm{span}(\boldsymbol{A})$, and $\mathrm{rank}(\boldsymbol{A})$ denote the column-span, and the rank of a matrix $\boldsymbol{A}$, respectively. An $n\times n$ identity matrix is denoted by $\boldsymbol{I}_n$ and an $a\times b$ zero matrix by $\mathbf{0}_{a\times b}$. Furthermore, let $(a)^+ = \max\{0,a\}$, for $a\in\mathbb{R}$. We will use distinct $i,j,k\in\mathcal{K}$ for the set of user indices $\mathcal{K} := \{1,2,3\}$ unless otherwise stated.

II. System Model

The MIMO 3-way channel comprises three full-duplex[1] users T_i with user-indices i in the set $\mathcal{K}=\{1,2,3\}$. A message from T_i to T_j is denoted by W_{ji} and has rate R_{ji} for $i\neq j\in\mathcal{K}$. Each user T_i desires to communicate a message to the two other users T_j and T_k. User T_i is equipped with $M_{\mathrm{T}}\in\mathbb{N}$ transmit antennas and with $M_{\mathrm{R}}\in\mathbb{N}$ receive antennas.

The signal transmitted at time-instant n from T_i is a vector $\boldsymbol{x}_i(n)\in\mathbb{C}^{M_{\mathrm{T}}\times 1}$ satisfying a power constraint P. The channel matrix for the MIMO channel from T_i to T_j is denoted $\boldsymbol{H}_{ji}\in\mathbb{C}^{M_{\mathrm{R}}\times M_{\mathrm{T}}}$ and i.i.d. randomly generated from a continuous probability distribution. The coefficients are assumed to be constant throughout the whole duration of the transmission. The received signal at T_j is a vector $\boldsymbol{y}_j(n)\in\mathbb{C}^{M_{\mathrm{R}}\times 1}$. $\boldsymbol{y}_j(n)$ is a superposition of the transmitted signals from T_i and T_k, weighted by $\boldsymbol{H}_{ji}$ and $\boldsymbol{H}_{jk}$, respectively, and by i.i.d. complex additive white Gaussian noise $\boldsymbol{z}_j\sim\mathcal{CN}(\mathbf{0}_{M_{\mathrm{R}}\times 1},\boldsymbol{I}_{M_{\mathrm{R}}})$:

$$\boldsymbol{y}_j(n)=\boldsymbol{H}_{ji}\boldsymbol{x}_i(n)+\boldsymbol{H}_{jk}\boldsymbol{x}_k(n)+\boldsymbol{z}_j(n). \tag{1}$$

After receiving $\boldsymbol{y}_j(n)$, T_j constructs $\boldsymbol{x}_j(n+1)$ as:

$$\boldsymbol{x}_j(n+1)=\mathcal{E}_{j,n}(W_{ij},W_{kj},\boldsymbol{y}_j^n), \tag{2}$$

where $\mathcal{E}_{j,n}$ is the encoding function of T_j at time-instant n, and sends $\boldsymbol{x}_j(n+1)$ in the next transmission. After N transmissions, where N is the length of one transmission block (codeword), T_j decodes W_{ji} and W_{jk} as follows:

$$(W_{ji},W_{jk})=\mathcal{D}_j(W_{ij},W_{kj},\boldsymbol{y}_j^N), \tag{3}$$

where $\mathcal{D}_j$ is the decoding function of T_j. All channel matrices are perfectly known at each user. Henceforth, we will neglect the time-instant n for notational simplicity unless necessary.

[1]We assume perfect full-duplex operation, and hence, there is no residual loop-back self-interference at each receiving user T_i.

Since we investigate the DoF [6] of this network, we define the DoF of a message W_{ji} by:

$$d_{ji}=\lim_{P\to\infty}\frac{R_{ji}}{\log(P)}. \tag{4}$$

The sum-DoF are computed by:

$$d_{\Sigma}=d_{12}+d_{21}+d_{13}+d_{31}+d_{32}+d_{23}. \tag{5}$$

III. Converse

Cut-set bounds: We begin with considering the cut-set bounds for the MIMO 3-way channel:

$$d_{ji}+d_{ki}\leq\min\{M_{\mathrm{T}},2M_{\mathrm{R}}\}, \tag{6}$$

$$d_{ij}+d_{ik}\leq\min\{2M_{\mathrm{T}},M_{\mathrm{R}}\}. \tag{7}$$

The right-hand side of (6) is the rank of the MIMO channel between T_i and a receiver formed by enabling full cooperation between T_j and T_k, with channel matrix $[\boldsymbol{H}_{ji}^{\mathsf{T}}\boldsymbol{H}_{ki}^{\mathsf{T}}]^{\mathsf{T}}$. A similar interpretation holds for the second bound. Combining (6) and (7) provides the sum-DoF bound:

$$d_{\Sigma}\leq\min\{3M_{\mathrm{T}},3M_{\mathrm{R}}\}. \tag{8}$$

Genie-aided bounds: We first assume that $M_{\mathrm{R}}\geq M_{\mathrm{T}}$. Assume every node can obtain its dedicated messages with an arbitrary small probability of error. Hence, T_2 can decode W_{21},W_{23} reliably from its available information, i. e., from $\boldsymbol{y}_2^N$, W_{12} and W_{32}, as shown in (3). Furthermore, we provide W_{31} to T_2 as side-information. We also provide T_2 with the correction-noise signal:

$$\tilde{\boldsymbol{z}}_2^N=\boldsymbol{z}_1^N-\boldsymbol{H}_{13}\boldsymbol{H}_{23}^{\dagger}\boldsymbol{z}_2^N, \tag{9}$$

as side-information[2]. Now, T_2 knows its decoded W_{21} and W_{31} by side-information. With W_{21}, W_{31}, user T_2 can generate $\boldsymbol{x}_1(1)$. By subtracting $\boldsymbol{H}_{21}\boldsymbol{x}_1(1)$ from $\boldsymbol{y}_2(1)$, and multiplying the result with $\boldsymbol{H}_{23}^{\dagger}$, T_2 can recover a noisy observation of $\boldsymbol{x}_3(1)$ given by $\boldsymbol{x}_3(1)+\boldsymbol{H}_{23}^{\dagger}\boldsymbol{z}_2(1)$. Next, T_2 multiplies this noisy observation by $\boldsymbol{H}_{13}$, and adds $\boldsymbol{H}_{12}\boldsymbol{x}_2(1)$ and $\tilde{\boldsymbol{z}}_2(1)$ to it to obtain $\boldsymbol{y}_1(1)$. Thus, T_2 obtains the first instance of $\boldsymbol{y}_1^N$. Knowing $\boldsymbol{y}_1(1)$, W_{21} and W_{31}, T_2 can generate $\boldsymbol{x}_1(2)$ (cf. equation (2)). Using $\boldsymbol{x}_1(2)$ again with $\boldsymbol{y}_2(2)$, T_2 can generate $\boldsymbol{y}_1(2)$ and $\boldsymbol{x}_1(3)$. T_2 proceeds this way until all instances (up to the N-th instance) of $\boldsymbol{y}_1^N$ have been generated. Now, having $\boldsymbol{y}_1^N$, W_{21}, and W_{31}, i. e., the same information as T_1, T_2 can decode W_{13} (cf. equation (3)). Therefore, given W_{31} and $\tilde{\boldsymbol{z}}_2^N$ as side-information, T_2 can decode W_{21}, W_{23} and W_{13}. Hence, the DoF of these messages are almost surely upper bounded by:

$$d_{21}+d_{23}+d_{13}\leq\mathrm{rank}([\boldsymbol{H}_{21}\,\boldsymbol{H}_{23}]) \tag{10}$$

$$=\min\{M_{\mathrm{R}},2M_{\mathrm{T}}\}. \tag{11}$$

We can apply a similar approach to bound:

$$d_{31}+d_{32}+d_{12}\leq\mathrm{rank}([\boldsymbol{H}_{31}\,\boldsymbol{H}_{32}]) \tag{12}$$

$$=\min\{M_{\mathrm{R}},2M_{\mathrm{T}}\}, \tag{13}$$

[2]The pseudo-inverse $\boldsymbol{H}_{23}^{\dagger}$ exists almost surely, since $\boldsymbol{H}_{23}$ is an $M_{\mathrm{R}}\times M_{\mathrm{T}}$ matrix with $M_{\mathrm{R}}\geq M_{\mathrm{T}}$.

by providing W_{21} and the correction noise-signal:

$$\tilde{\boldsymbol{z}}_3^N = \boldsymbol{z}_1^N - \boldsymbol{H}_{12}\boldsymbol{H}_{32}^{\dagger}\boldsymbol{z}_3^N \tag{14}$$

to T_3. As a result, T_3 can construct $\boldsymbol{y}_1^N$ and decode W_{12} reliably. Combining (11) and (13), bounds the sum-DoF to:

$$d_\Sigma \le \min\{2M_\mathrm{R}, 4M_\mathrm{T}\}. \tag{15}$$

For the contrary case, we assume that $M_\mathrm{R} < M_\mathrm{T}$ holds. We enhance the number of receive antennas at all receivers to $\mathring{M}_\mathrm{R} = M_\mathrm{T}$. The effective channel output at T_3 becomes:

$$\mathring{\boldsymbol{y}}_3(n) = \mathring{\boldsymbol{H}}_{31}\boldsymbol{x}_1(n) + \mathring{\boldsymbol{H}}_{32}\boldsymbol{x}_2(n) + \mathring{\boldsymbol{z}}_3(n), \tag{16}$$

with the extended $M_\mathrm{T} \times M_\mathrm{T}$ matrices[3] $\mathring{\boldsymbol{H}}_{31}$, $\mathring{\boldsymbol{H}}_{32}$, and the extended $M_\mathrm{T} \times 1$ noise vector $\mathring{\boldsymbol{z}}_3(n)$. We can apply the upper bounds derived in (11) and (13) now, leading to:

$$d_{21} + d_{23} + d_{13} \le M_\mathrm{T}, \tag{17}$$
$$d_{31} + d_{32} + d_{12} \le M_\mathrm{T}, \tag{18}$$
$$d_\Sigma \le 2M_\mathrm{T}. \tag{19}$$

Combining these bounds with the cut-set bounds yields:

$$\begin{aligned} d_\Sigma &\le \min\{2M_\mathrm{R}, 4M_\mathrm{T}, 3M_\mathrm{R}, 3M_\mathrm{T}\} \\ &= \min\{2M_\mathrm{R}, 3M_\mathrm{T}\}, \text{ if } M_\mathrm{T} \le M_\mathrm{R}, \qquad (20) \\ d_\Sigma &\le \min\{2M_\mathrm{T}, 3M_\mathrm{R}, 3M_\mathrm{T}\} \\ &= \min\{2M_\mathrm{T}, 3M_\mathrm{R}\}, \text{ if } M_\mathrm{T} > M_\mathrm{R}. \qquad (21) \end{aligned}$$

Theorem 1. *The DoF of the MIMO 3-way channel with M_T transmit and M_R receive antennas at each user T_i are:*

$$d_\Sigma = \begin{cases} \min\{2M_\mathrm{R}, 3M_\mathrm{T}\}, & \text{if } M_\mathrm{T} \le M_\mathrm{R}, \\ \min\{2M_\mathrm{T}, 3M_\mathrm{R}\}, & \text{if } M_\mathrm{T} > M_\mathrm{R}. \end{cases} \tag{22}$$

IV. Achievability

The following communication schemes provide achievability of the upper bounds in Theorem 1. Note that, symbol-extensions over multiple time-slots are used on constant MIMO channels to achieve non-integer DoF per user, cf. [7], [11].

A. Case $M_\mathrm{T} \le M_\mathrm{R}$ with $d_\Sigma = 3M_\mathrm{T}$

The dominant term in (22) yields $3M_\mathrm{T}$ if $3M_\mathrm{T} \le 2M_\mathrm{R}$ holds. We use the following symmetric DoF allocation:

$$d := d_{ij} = d_{ji}. \tag{23}$$

We further decompose the symmetric DoF d further for IA (tilde-notation, $\tilde{d}$) and for beam-forming (bar-notation, $\bar{d}$):

$$\bar{d} := \bar{d}_{ij} = \bar{d}_{ji}, \tag{24}$$
$$\tilde{d} := \tilde{d}_{ij} = \tilde{d}_{ji}, \tag{25}$$
$$d := \bar{d} + \tilde{d}. \tag{26}$$

[3]The inverses exist almost surely.

In other words, we demand that bidirectional signals pairwise occupy the same number of DoF. According to the assumptions on M_T and M_R, the following bounds must hold:

$$0 \le 2d \le M_\mathrm{T}, \tag{27}$$
$$0 \le 3d \le \min\{M_\mathrm{R}, 2M_\mathrm{T}\}, \tag{28}$$

so that all upper bounds provided in Section III are satisfied.

Messages W_{ji} are encoded into complex-valued symbol streams $\tilde{\boldsymbol{u}}_{ji} \in \mathbb{C}^{\tilde{d}\times 1}$ and $\bar{\boldsymbol{u}}_{ji} \in \mathbb{C}^{\bar{d}\times 1}$. These symbol streams are pre-coded at the transmitters and post-coded at the receivers, so that the proposed sum-DoF are achieved. For precoding, we use beam-forming matrices $\tilde{\boldsymbol{V}}_{ji} \in \mathbb{C}^{M_\mathrm{T}\times\tilde{d}}$ and $\bar{\boldsymbol{V}}_{ji} \in \mathbb{C}^{M_\mathrm{T}\times\bar{d}}$. Transmit signals $\boldsymbol{x}_i$ are constructed from the pre-coded symbol streams as:

$$\boldsymbol{x}_i = [\tilde{\boldsymbol{V}}_{ji}\ \bar{\boldsymbol{V}}_{ji}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{ji}\\ \bar{\boldsymbol{u}}_{ji}\end{smallmatrix}\right] + [\tilde{\boldsymbol{V}}_{ki}\ \bar{\boldsymbol{V}}_{ki}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{ki}\\ \bar{\boldsymbol{u}}_{ki}\end{smallmatrix}\right]. \tag{29}$$

First, we consider the intersection space of the two incident subchannels at the receiver of T_j. The number of dimensions for $\operatorname{span}(\boldsymbol{H}_{ji}) \cap \operatorname{span}(\boldsymbol{H}_{jk})$ is computed by Lemma 1 as given in the appendix. It has $0 \le (2M_\mathrm{T} - M_\mathrm{R})^+ \le \frac{1}{3}M_\mathrm{R}$ dimensions, since $3M_\mathrm{T} \le 2M_\mathrm{R}$. We fix

$$\tilde{d} = (2M_\mathrm{T} - M_\mathrm{R})^+ \tag{30}$$

and design $\tilde{\boldsymbol{V}}_{ji}$ such that the two dedicated signals remain distinct, while interference is aligned at each receiver:

$$\operatorname{span}(\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}) = \operatorname{span}(\boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}). \tag{31}$$

After this first part of pre-coding, a number of:

$$\bar{M}_\mathrm{T} = M_\mathrm{T} - 2\tilde{d} \ge 0, \tag{32}$$
$$\bar{M}_\mathrm{R} = M_\mathrm{R} - 3\tilde{d} \ge 0, \tag{33}$$

transmit and receive dimensions remains available at each user, respectively. Since the complete intersection space is already consumed by IA, we have $(2\bar{M}_\mathrm{T} - \bar{M}_\mathrm{R})^+ = 0$ and hence, $2\bar{M}_\mathrm{T} \le \bar{M}_\mathrm{R}$ holds. The remaining DoF are allocated by:

$$\bar{d} = \tfrac{1}{2}\bar{M}_\mathrm{T}, \tag{34}$$

for all $k \in \mathcal{K}$. The beam-forming matrices $\bar{\boldsymbol{V}}_{ji}$ and $\bar{\boldsymbol{V}}_{jk}$ are chosen such that the two signals $\boldsymbol{x}_i$ and $\boldsymbol{x}_k$ are received linearly independent at T_j. This allocation satisfies both upper bounds. The received signals at T_j yield:

$$\begin{aligned}\boldsymbol{y}_j = \Big(&\boldsymbol{H}_{ji}[\tilde{\boldsymbol{V}}_{ji}\ \bar{\boldsymbol{V}}_{ji}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{ji}\\ \bar{\boldsymbol{u}}_{ji}\end{smallmatrix}\right] + \boldsymbol{H}_{jk}[\tilde{\boldsymbol{V}}_{jk}\ \bar{\boldsymbol{V}}_{jk}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{jk}\\ \bar{\boldsymbol{u}}_{jk}\end{smallmatrix}\right]\Big) + \\ &\Big(\boldsymbol{H}_{ji}[\tilde{\boldsymbol{V}}_{ki}\ \bar{\boldsymbol{V}}_{ki}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{ki}\\ \bar{\boldsymbol{u}}_{ki}\end{smallmatrix}\right] + \boldsymbol{H}_{jk}[\tilde{\boldsymbol{V}}_{ik}\ \bar{\boldsymbol{V}}_{ik}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{ik}\\ \bar{\boldsymbol{u}}_{ik}\end{smallmatrix}\right]\Big) + \boldsymbol{z}_j.\end{aligned}$$

The first sum in brackets describes the dedicated signals. The second sum in brackets describes the interfering signals at T_j (with $\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}$ and $\boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}$ aligned). The signal and interference subspaces are linearly independent, since the composite $M_\mathrm{R} \times (3\tilde{d} + 4\bar{d})$ matrix

$$[\boldsymbol{H}_{ji}[\tilde{\boldsymbol{V}}_{ji}\ \bar{\boldsymbol{V}}_{ji}\ \tilde{\boldsymbol{V}}_{ki}\ \bar{\boldsymbol{V}}_{ki}]\,\boldsymbol{H}_{jk}[\tilde{\boldsymbol{V}}_{jk}\ \bar{\boldsymbol{V}}_{jk}\ \bar{\boldsymbol{V}}_{ik}]], \tag{35}$$

has full column rank, almost surely, due to:

$$3\tilde{d} + 4\bar{d} = 2M_\mathrm{T} - (2M_\mathrm{T} - M_\mathrm{R})^+ < M_\mathrm{R}. \tag{36}$$

In the post-coding step, each receiver T_j uses a composite zero-forcing matrix $\boldsymbol{N}_j = [\boldsymbol{N}_{ji}^{\mathsf{T}} \boldsymbol{N}_{jk}^{\mathsf{T}}]^{\mathsf{T}}$ to separate and decode its two dedicated signals and to eliminate the interfering signals. The received signals $\boldsymbol{y}_j$ are filtered by the two corresponding zero-forcing matrices $\boldsymbol{N}_{ji}, \boldsymbol{N}_{jk} \in \mathbb{C}^{d\times M_\mathrm{R}}$, with:

$$\boldsymbol{N}_{ji}[\boldsymbol{H}_{jk}(\tilde{\boldsymbol{V}}_{jk}+\bar{\boldsymbol{V}}_{jk}+\tilde{\boldsymbol{V}}_{ik}+\bar{\boldsymbol{V}}_{ik})+\boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ki}]=\boldsymbol{0}_{d\times(2d+\bar{d})}, \quad (37)$$

$$\boldsymbol{N}_{jk}[\boldsymbol{H}_{ji}(\tilde{\boldsymbol{V}}_{ji}+\bar{\boldsymbol{V}}_{ji}+\tilde{\boldsymbol{V}}_{ki}+\bar{\boldsymbol{V}}_{ki})+\boldsymbol{H}_{jk}\bar{\boldsymbol{V}}_{ik}]=\boldsymbol{0}_{d\times(2d+\bar{d})}, \quad (38)$$

so that filtering with $\boldsymbol{N}_{ji}\boldsymbol{y}_j$ and $\boldsymbol{N}_{jk}\boldsymbol{y}_j$ provides $d_{ji}+d_{jk} = 2d$ noisy interference-free streams of dedicated signals at T_j:

$$\boldsymbol{N}_{ji}\boldsymbol{y}_j = \boldsymbol{N}_{ji}\boldsymbol{H}_{ji}\left[\tilde{\boldsymbol{V}}_{ji}\tilde{\boldsymbol{u}}_{ji} + \bar{\boldsymbol{V}}_{ji}\bar{\boldsymbol{u}}_{ji}\right] + \boldsymbol{N}_{ji}\boldsymbol{z}_j, \quad (39)$$

$$\boldsymbol{N}_{jk}\boldsymbol{y}_j = \boldsymbol{N}_{jk}\boldsymbol{H}_{jk}\left[\tilde{\boldsymbol{V}}_{jk}\tilde{\boldsymbol{u}}_{jk} + \bar{\boldsymbol{V}}_{jk}\bar{\boldsymbol{u}}_{jk}\right] + \boldsymbol{N}_{jk}\boldsymbol{z}_j. \quad (40)$$

Thus, each user T_j can decode its two dedicated streams with:

$$d_{ji} + d_{jk} = 2d = 2(\tilde{d} + \bar{d}) = M_\mathrm{T}. \quad (41)$$

Altogether, $3M_\mathrm{T}$ DoF in the first term of (22) are achieved:

$$d_\Sigma \le 6d = 3M_\mathrm{T}.$$

B. Case $M_\mathrm{T} \le M_\mathrm{R}$ with $d_\Sigma = 2M_\mathrm{R}$

On the other hand, the dominant term in (22) yields $2M_\mathrm{R}$ if $2M_\mathrm{R} \le 3M_\mathrm{T}$ holds. Then, $\mathrm{span}(\boldsymbol{H}_{ji}) \cap \mathrm{span}(\boldsymbol{H}_{jk})$ at T_j has $2M_\mathrm{T} - M_\mathrm{R} > \frac{1}{3}M_\mathrm{R}$ dimensions. We allocate:

$$d = \tilde{d} = \tfrac{1}{3}M_\mathrm{R}. \quad (42)$$

for all $i \in \mathcal{K}$. This allocation satisfies all upper bounds:

$$0 \le 2d \le M_\mathrm{T}, \quad (43)$$

$$0 \le 3d \le M_\mathrm{R}, \quad (44)$$

and no remaining dimensions are left at the receivers. The symbol streams $\tilde{\boldsymbol{u}}_{ji} \in \mathbb{C}^{\tilde{d}\times 1}$ are pre-coded by the beam-forming matrices $\tilde{\boldsymbol{V}}_{ji} \in \mathbb{C}^{M_\mathrm{T}\times\tilde{d}}$ and aligned analogously to (31). Hence, the received signal at T_j is:

$$\boldsymbol{y}_j = \left(\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ji}\tilde{\boldsymbol{u}}_{ji} + \boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{jk}\tilde{\boldsymbol{u}}_{jk}\right) + \left(\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}\tilde{\boldsymbol{u}}_{ki} + \boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}\tilde{\boldsymbol{u}}_{ik}\right) + \boldsymbol{z}_j. \quad (45)$$

The composite $M_\mathrm{R} \times 3\tilde{d}$-dimensional matrix:

$$[\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ji} \ \boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{jk} \ \boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}], \quad (46)$$

has full column rank, almost surely, so that dedicated and interfering signals are linearly independent.

For post-coding, the zero-forcing matrices are chosen as:

$$\boldsymbol{N}_{ji}(\boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{jk} + \boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}) = \boldsymbol{0}_{\tilde{d}\times 2\tilde{d}}, \quad (47)$$

$$\boldsymbol{N}_{jk}(\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ji} + \boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}) = \boldsymbol{0}_{\tilde{d}\times 2\tilde{d}}, \quad (48)$$

such that the filtered signals yield:

$$\boldsymbol{N}_{ji}\boldsymbol{y}_j = \boldsymbol{N}_{ji}\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ji}\tilde{\boldsymbol{u}}_{ji} + \boldsymbol{N}_{ji}\boldsymbol{z}_j, \quad (49)$$

$$\boldsymbol{N}_{jk}\boldsymbol{y}_j = \boldsymbol{N}_{jk}\boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{jk}\tilde{\boldsymbol{u}}_{jk} + \boldsymbol{N}_{jk}\boldsymbol{z}_j. \quad (50)$$

Each user T_j can decode noisy but interference-free versions of its two dedicated streams and achieves:

$$d_{ji} + d_{jk} = 2\tilde{d} = \tfrac{2}{3}M_\mathrm{R}, \quad (51)$$

so that the sum-DoF are:

$$d_\Sigma = 6d = 2M_\mathrm{R}. \quad (52)$$

Thence, the upper bound $\min\{2M_\mathrm{R}, 3M_\mathrm{T}\}$ is shown to be achievable.

C. Case $M_\mathrm{T} > M_\mathrm{R}$ with $d_\Sigma = 2M_\mathrm{T}$

The upper bound (22) yields $2M_\mathrm{T}$, if $2M_\mathrm{T} \le 3M_\mathrm{R}$ holds. Again we use the symmetric DoF allocation as defined in (23) to (26). In this case, the following upper bounds must hold:

$$0 \le 2d \le M_\mathrm{R}, \quad (53)$$

$$0 \le 3d \le M_\mathrm{T}. \quad (54)$$

Since $M_\mathrm{T} > M_\mathrm{R}$, zero-forcing beam-forming as in [11] and [12] is applicable, we allocate:

$$\bar{d} = M_\mathrm{T} - M_\mathrm{R} < \tfrac{1}{2}M_\mathrm{R} \quad (55)$$

dimensions. Analogous to Section IV-A, we pre-code the symbol-streams $\bar{\boldsymbol{u}}_{ji}$ and $\tilde{\boldsymbol{u}}_{ji}$ to construct the transmit signal $\boldsymbol{x}_i$ as in (29). The beam-forming matrix $\bar{\boldsymbol{V}}_{ki}$ has $M_\mathrm{T} \times \bar{d}$ dimensions and is designed to cast the interfering signal into the $(M_\mathrm{T} - M_\mathrm{R})$-dimensional null-space of T_j:

$$\boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ki} = \boldsymbol{0}_{M_\mathrm{R}\times\bar{d}}. \quad (56)$$

For the next step, the number of remaining transmit and receive dimensions per user available for IA are:

$$\tilde{M}_\mathrm{T} = M_\mathrm{T} - 2(M_\mathrm{T} - M_\mathrm{R}) = 2M_\mathrm{R} - M_\mathrm{T}, \quad (57)$$

$$\tilde{M}_\mathrm{R} = M_\mathrm{R} - 2(M_\mathrm{T} - M_\mathrm{R}) = 3M_\mathrm{R} - 2M_\mathrm{T}. \quad (58)$$

Since $2\tilde{M}_\mathrm{T} > \tilde{M}_\mathrm{R}$ holds, the remaining dimensions suffice for IA. Furthermore, since $3\tilde{M}_\mathrm{T} > 2\tilde{M}_\mathrm{R}$, more than $\frac{1}{3}\tilde{M}_\mathrm{R}$ dimensions are available for IA between each user pair (cf. Lemma 1). To establish a fair scheme, we set:

$$\tilde{d} = \tfrac{1}{3}\tilde{M}_\mathrm{R} = M_\mathrm{R} - \tfrac{2}{3}M_\mathrm{T}. \quad (59)$$

The beam-forming matrices $\tilde{\boldsymbol{V}}_{ki}$ and $\tilde{\boldsymbol{V}}_{ik}$, each with $M_\mathrm{T} \times \tilde{d}$ dimensions, are chosen such that the bidirectional interference signals are aligned at receiver T_j, as analogously done in (31). Due to zero-forcing beam-forming, the symbol streams $\bar{\boldsymbol{u}}_{ki}$ and $\bar{\boldsymbol{u}}_{ik}$ are not received at T_j, so that we obtain:

$$\boldsymbol{y}_j = \left(\boldsymbol{H}_{ji}[\tilde{\boldsymbol{V}}_{ji}\bar{\boldsymbol{V}}_{ji}]\begin{bmatrix}\tilde{\boldsymbol{u}}_{ji}\\ \bar{\boldsymbol{u}}_{ji}\end{bmatrix} + \boldsymbol{H}_{jk}[\tilde{\boldsymbol{V}}_{jk}\bar{\boldsymbol{V}}_{jk}]\begin{bmatrix}\tilde{\boldsymbol{u}}_{jk}\\ \bar{\boldsymbol{u}}_{jk}\end{bmatrix}\right) + \left(\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}\tilde{\boldsymbol{u}}_{ki} + \boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}\tilde{\boldsymbol{u}}_{ik}\right) + \boldsymbol{z}_j. \quad (60)$$

The signal and interference subspaces are linearly independent, almost surely, since the composite matrix:

$$[\boldsymbol{H}_{ji}[\tilde{\boldsymbol{V}}_{ji} \ \bar{\boldsymbol{V}}_{ji} \ \tilde{\boldsymbol{V}}_{ki}] \ \boldsymbol{H}_{jk}[\tilde{\boldsymbol{V}}_{jk} \ \bar{\boldsymbol{V}}_{jk}]], \quad (61)$$

of $M_\mathrm{R} \times (3\tilde{d} + 2\bar{d})$ dimensions ($\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}$ and $\boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}$ are aligned) has full column rank.

For post-coding at the receivers, we use zero-forcing matrices $\boldsymbol{N}_{ji}$ of $d \times M_\mathrm{R}$ dimensions as given in (37) and (38), but for differently allocated d according to (60). Analogously, the following signals are obtained after filtering:

$$\boldsymbol{N}_{ji}\boldsymbol{y}_j = \boldsymbol{N}_{ji}\boldsymbol{H}_{ji}\left[\tilde{\boldsymbol{V}}_{ji}\tilde{\boldsymbol{u}}_{ji} + \bar{\boldsymbol{V}}_{ji}\bar{\boldsymbol{u}}_{ji}\right] + \boldsymbol{N}_{ji}\boldsymbol{z}_j,$$
$$\boldsymbol{N}_{jk}\boldsymbol{y}_j = \boldsymbol{N}_{jk}\boldsymbol{H}_{jk}\left[\tilde{\boldsymbol{V}}_{jk}\tilde{\boldsymbol{u}}_{jk} + \bar{\boldsymbol{V}}_{jk}\bar{\boldsymbol{u}}_{jk}\right] + \boldsymbol{N}_{jk}\boldsymbol{z}_j.$$

T_j decodes two noisy but interference-free dedicated streams:

$$d_{ji} + d_{jk} = 2d = 2(\tilde{d} + \bar{d}) = \tfrac{2}{3}M_\mathrm{T}, \quad (62)$$

so that the sum-DoF of $2M_\mathrm{T}$ are achieved:

$$d_\Sigma = 6d = 2M_\mathrm{T}. \quad (63)$$

D. Case $M_\mathrm{T} > M_\mathrm{R}$ with $d_\Sigma = 3M_\mathrm{R}$

In the case $2M_\mathrm{T} \geq 3M_\mathrm{R}$, the upper bound (22) yields $3M_\mathrm{R}$. Now it suffices to use zero-forcing beam-forming only. IA is actually not necessary for this case. We allocate the DoF:

$$d = \bar{d} = \tfrac{1}{2}M_\mathrm{R} \quad (64)$$

for all $k \in \mathcal{K}$, satisfying (53) and (54). We use the beam-forming matrices $\bar{\boldsymbol{V}}_{ji}$ with $M_\mathrm{T} \times \bar{d}$ dimensions and cast interference to the null-space of the undesired receivers:

$$\boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ki} = \boldsymbol{0}_{M_\mathrm{R}\times\bar{d}}. \quad (65)$$

The received signal at receiver T_j is:

$$\boldsymbol{y}_j = \boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ji}\bar{\boldsymbol{u}}_{ji} + \boldsymbol{H}_{jk}\bar{\boldsymbol{V}}_{jk}\bar{\boldsymbol{u}}_{jk} + \boldsymbol{z}_j. \quad (66)$$

The dedicated signals are linearly independent, almost surely, since the composite $M_\mathrm{R} \times 2\bar{d}$ matrix has full column rank:

$$[\boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ji} \ \boldsymbol{H}_{jk}\bar{\boldsymbol{V}}_{jk}]. \quad (67)$$

For post-coding at receiver T_j, we use the zero-forcing matrices $\boldsymbol{N}_{ji}$ and $\boldsymbol{N}_{jk}$ of $\bar{d} \times M_\mathrm{R}$ dimensions each so that (47) and (48) hold. We obtain the following filtered signals:

$$\boldsymbol{N}_{ji}\boldsymbol{y}_j = \boldsymbol{N}_{ji}\boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ji}\bar{\boldsymbol{u}}_{ji} + \boldsymbol{N}_{ji}\boldsymbol{z}_j, \quad (68)$$
$$\boldsymbol{N}_{jk}\boldsymbol{y}_j = \boldsymbol{N}_{jk}\boldsymbol{H}_{jk}\bar{\boldsymbol{V}}_{jk}\bar{\boldsymbol{u}}_{jk} + \boldsymbol{N}_{jk}\boldsymbol{z}_j. \quad (69)$$

Each receiver T_j can decode:

$$d_{ji} + d_{jk} = 2d = 2\bar{d} = M_\mathrm{R}, \quad (70)$$

and achieves the sum-DoF of:

$$d_\Sigma = 6d = 3M_\mathrm{R}. \quad (71)$$

As $\min\{2M_\mathrm{T}, 3M_\mathrm{R}\}$ is also shown to be achievable, the proof of Theorem 1 is concluded now. Furthermore, complete fairness is always maintained among all users.

V. Symmetry

The parameter plane of the symmetric DoF depicted in Figure 2 provides a symmetry along the intersecting line $M_\mathrm{T} = M_\mathrm{R}$ for all parameters M_T and M_R. At that line, the antenna parameters of the achieved DoF are swapped since null-space beam-forming and linear independent beam-forming are swapped.

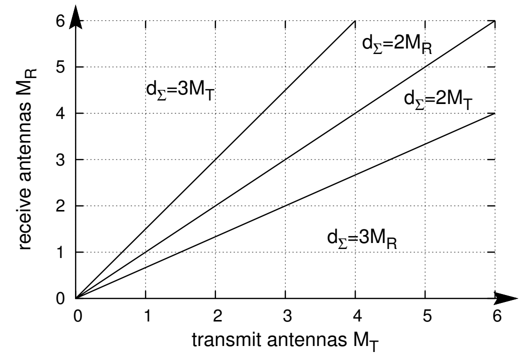

Fig. 2. The parameter plane with $0 \leq M_\mathrm{T} \leq 6$ transmit antennas and $0 \leq M_\mathrm{R} \leq 6$ receive antennas of the sum-DoF d_Σ is partitioned into four sectors.

Appendix

Lemma 1. *If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are both complex $M_\mathrm{R} \times M_\mathrm{T}$ random matrices, respectively, whose entries are drawn randomly i. i. d., then there exists a $(2\min\{M_\mathrm{T}, M_\mathrm{R}\} - M_\mathrm{R})^+$-dimensional intersection subspace between the two column spaces of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, almost surely.*

The proof is similar to [12, Lem. 1] and omitted here. Note further that $\mathrm{rank}(\boldsymbol{A}_i) = \min(M_\mathrm{T}, M_\mathrm{R})$ holds for $i = 1, 2$.

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