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Excercise 1
- Proposed Solution -

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Solution of Problem 1

a) Division with remainder is computed as follows:

Algorithm 1 Division with remainder

input: Two integers, the dividend a and the divisor d with $a \geq d$

output: Integer division: a div d, and remainder: a mod d)

procedure DIVMOD(a, b)
    Find the unique $q \in \mathbb{N}$ such that $a = q \cdot d + r$ holds with $0 \leq r < d$
    return (q, r)
end procedure

Ordinary long division yields $1234 : 357 = 3 + \frac{163}{357} \approx 3.456$
We obtain $q = 1234 \text{ div } 357 = 3$, and $r = 1234 \mod 357 \equiv 163$

b) The greatest common divisor (gcd) is computed as follows:

Algorithm 2 Euclidean algorithm to compute the greatest common divisor

input: Two integers a and b with $a \geq b$

output: The greatest common divisor $\gcd(a, b)$

procedure GCD(a, b)
    while $b \neq 0$ **do**
        $r \leftarrow a \bmod b$
        $a \leftarrow b$
        $b \leftarrow r$
    end while
    return a
end procedure

To compute $\gcd(357, 1234)$, we can compactly write:

$$1234 = 357 \cdot 3 + 163$$
$$357 = 163 \cdot 2 + 31$$
$$163 = 31 \cdot 5 + 8$$
$$31 = 8 \cdot 3 + 7$$
$$8 = 7 \cdot 1 + 1$$
$$(7 = 1 \cdot 7 + 0) \quad //\text{but, } r = 0 ↯$$

Hence, we obtain $\gcd(357, 1234) = 1$.

c) The extended Euclidean algorithm (EEA) is used to compute multiplicative inverses:

Algorithm 3 Extended Euclidean algorithm

input: Two integers a and b with $a \geq b$

output: An integer tuple (u, d, v) satifying $a \cdot u + b \cdot v = d = \gcd(a, b)$

```
procedure EXTGCD(a, b)
    u ← 1
    v ← 0
    d ← a
    v_1 ← 0
    v_3 ← b
    while v_3 ≠ 0 do
        q ← ⌊d / v_3⌋
        t_3 ← d mod v_3
        t_1 ← u − q · v_1
        u ← v_1
        d ← v_3
        v_1 ← t_1
        v_3 ← t_3
        return a
    end while
    v ← (d − a·u) / b
    return (u, d, v)
end procedure
```

A compact computation of the inverse using this algorithm is, e.g.:

$$
\begin{aligned}
1 &= 8 - 7 \cdot 1 \checkmark \\
&= 8 - (31 - 8 \cdot 3) \cdot 1 \\
&= 8 \cdot 4 - 31 \cdot 1 \checkmark \\
&= (163 - 31 \cdot 5) \cdot 4 - 31 \cdot 1 \\
&= 163 \cdot 4 - 31 \cdot 21 \checkmark \\
&= 163 \cdot 4 - (357 - 163 \cdot 2) \cdot 21 \\
&= 163 \cdot 46 - 357 \cdot 21 \checkmark \\
&= (1234 - 357 \cdot 3) \cdot 46 - 357 \cdot 21 \\
&= 1234 \cdot 46 - 357 \cdot 159 \checkmark
\end{aligned}
$$

Thus, the multiplicative inverse to 357 is -159 modulo 1234.

d) We consider the two polynomials $b(x) = x^3 + x + 1$ and $m(x) = x^5 + x^3 + 1$.

Since the coefficients are in $\{0, 1\}$, addition and subtraction is equivalent here.

We compute $\gcd(b(x), m(x))$ using polynomial division:

$$
\begin{array}{l}
(x^5 + x^3 + 1) : (x^3 + x + 1) = x^2 + \frac{x^2+1}{x^3+x+1} \\
-(x^5 + x^3 + x^2) \\
\hline
0 + 0 + x^2 + 1
\end{array}
$$

This yields the first step of the Euclidean algorithm:

$x^5 + x^3 + 1 = (x^3 + x + 1) \cdot x^2 + (x^2 + 1)$

In the second step of the Euclidean Algorithm, we again use polynomial division:

$$\begin{array}{l} (x^3 + x + 1) : (x^2 + 1) = x + \frac{1}{x^2+1} \\ -(x^3 + x) \\ \hline 0 + 1 \end{array}$$

This yields $x^3 + x + 1 = (x^2 + 1) \cdot x + 1$, so that $\gcd(x^5 + x^3 + 1, x^3 + x + 1) = 1$

Applying the extended Euclidean algorithm to these polynomials yields:

$$\begin{aligned} 1 &= (x^3 + x + 1) + x(x^2 + 1) \\ &= (x^3 + x + 1) + x[(x^5 + x^3 + 1) + x^2(x^3 + x + 1)] \\ &= (x^3 + x + 1)(1 + x^3) + x(x^5 + x^3 + 1) \end{aligned}$$

Thus, the multiplicative inverse to $b(x) = x^3 + x + 1$ is $a(x) = b^{-1}(x) = x^3 + 1$

Solution of Problem 2

a) Show that from $a \,|\, b$ and $b \,|\, c$ it follows that $a \,|\, c$.
$a|b \Rightarrow \exists k_1 \in \mathbb{Z} : b = k_1 \cdot a$
$b|c \Rightarrow \exists k_2 \in \mathbb{Z} : c = k_2 \cdot b$
$\Rightarrow c = k_1 \cdot k_2 \cdot a$
$\Rightarrow k = k_1 \cdot k_2$
$\Rightarrow \exists k \in \mathbb{Z} : c = k \cdot a$
$\Rightarrow a|c$

b) Show that from $a \,|\, b$ and $c \,|\, d$ it follows that $(ac) \,|\, (bd)$.
$a|b \Rightarrow \exists k_1 \in \mathbb{Z} : b = k_1 \cdot a$
$c|d \Rightarrow \exists k_2 \in \mathbb{Z} : d = k_2 \cdot c$
$\Rightarrow b \cdot d = k_1 \cdot a \cdot k_2 \cdot c$
$\Rightarrow k = k_1 \cdot k_2$
$\Rightarrow \exists k \in \mathbb{Z} : b \cdot d = k \cdot a \cdot c$
$\Rightarrow (a \cdot c)|(b \cdot d)$

c) Show that from $a \,|\, b$ and $a \,|\, c$ it follows that $a \,|\, (xb + yc) \;\; \forall \, x, y \in \mathbb{Z}$.
$a|b \Rightarrow \exists k_1 \in \mathbb{Z} : b = k_1 \cdot a$
$\Rightarrow x \in \mathbb{Z}, x \cdot b = xk_1 \cdot a$
$a|c \Rightarrow \exists k_2 \in \mathbb{Z} : c = k_2 \cdot a$
$\Rightarrow y \in \mathbb{Z}, y \cdot c = yk_2 \cdot a$
$xb + yc = xk_1 \cdot a + yk_2 \cdot a = (xk_1 + yk_2)a$
$\Rightarrow k = xk_1 + yk_2$
$\Rightarrow \exists k \in \mathbb{Z} : (xb + yc) = k \cdot a$
$\Rightarrow a|(xb + yc)$