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Exercise 4
- Proposed Solution -

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Solution of Problem 1

Theorem 4.3 shall be proven.

a) X is a discrete random variable with $p_i = P(X = x_i)$, $i = 1, \ldots, m$. It holds

$$H(X) = -\sum_i p_i \log(p_i) \geq 0,$$

as $p_i \geq 0$ and $-\log(p_i) \geq 0$ for $0 < p_i \leq 1$ and $0 \cdot \log 0 = 0$ per definition. Equality holds, if all addends are zero, i.e.,

$$p_i \log(p_i) = 0 \Leftrightarrow p_i \in \{0, 1\} \quad i = 1, \ldots, m,$$

as $p_i > 0$ and $-\log(p_i) > 0$, thus, $-p_i \log(p_i) > 0$ for $0 < p_i < 1$.

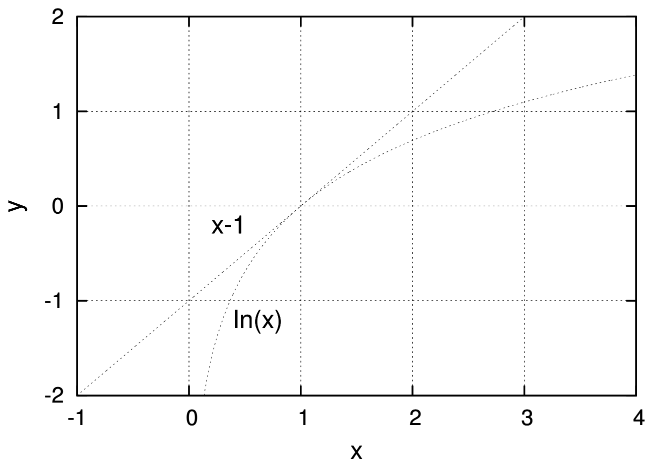
b) It holds

$$\begin{aligned}
H(X) - \log(m) &= -\sum_i p_i \log(p_i) - \underbrace{\sum_i p_i}_{=1} \log(m) \\
&= \sum_{i:p_i>0} p_i \log\left(\frac{1}{p_i\, m}\right) \\
&= (\log e) \sum_{i:p_i>0} p_i \ln\left(\frac{1}{p_i\, m}\right) \\
&\overset{\ln(x)\leq x-1}{\leq} (\log e) \sum_{i:p_i>0} p_i \left(\frac{1}{p_i\, m} - 1\right) \\
&= (\log e) \sum_{i:p_i>0} \left(\frac{1}{m} - p_i\right) = 0
\end{aligned}$$

As $\ln(x) = x - 1$ only holds for $x = 1$ it follows that equality holds iff $p_i = 1/m$, $i = 1, \ldots, m$. In particular, as $p_i = \frac{1}{m}$, it follows $p_i > 0$, $i = 1, \ldots, m$.

c) Define for $i = 1, \ldots, m$ and $j = 1, \ldots, d$

$$p_{i|j} = P(X = x_i \mid Y = y_j).$$



Show $H(X \mid Y) - H(X) \le 0$ which is equivalent to the claim.

$$
\begin{aligned}
H(X \mid Y) - H(X) &= -\sum_{i,j} p_{i,j} \log(p_{i|j}) + \sum_i p_i \log(p_i) \\
&= -\sum_{i,j} p_{i,j} \log\left(\frac{p_{i,j}}{p_j}\right) + \sum_i \underbrace{\sum_j p_{i,j}}_{=p_i} \log(p_i) \\
&= (\log e) \sum_{i,j: p_{i,j}>0} p_{i,j} \ln\left(\frac{p_i\, p_j}{p_{i,j}}\right) \\
&\overset{\ln(x)\le x-1}{\le} (\log e) \sum_{i,j: p_{i,j}>0} p_{i,j} \left(\frac{p_i\, p_j}{p_{i,j}} - 1\right) \\
&= (\log e) \sum_{i,j: p_{i,j}>0} (p_i\, p_j - p_{i,j}) = 0
\end{aligned}
$$

Note that from $p_{i,j} > 0$ it follows $p_i, p_j > 0$. Equality hold for $p_i\, p_j = p_{i,j}$ which is equivalent to X and Y being stochastically independent.

This means that the mutual information $I(X, Y) = H(X) - H(X \mid Y)$ is nonnegative.

d) It holds

$$
\begin{aligned}
H(X, Y) &= -\sum_{i,j} p_{i,j} \log(p_{i,j}) \\
&= -\sum_{i,j} p_{i,j} \left[\log(p_{i,j}) - \log(p_i) + \log(p_i)\right] \\
&= -\sum_{i,j} p_{i,j} \log \underbrace{\left(\frac{p_{i,j}}{p_i}\right)}_{p_{j|i}} - \sum_i \underbrace{\sum_j p_{i,j}}_{=p_i} \log(p_i) \\
&= H(Y \mid X) + H(X).
\end{aligned}
$$

e) It holds

$$
H(X, Y) \overset{(d)}{=} H(X) + H(Y \mid X) \overset{(c)}{\le} H(X) + H(Y)
$$

with equality as in (c) iff X and Y are stochastically independent.

Solution of Problem 2

Recall: $H(X) = -\sum_i p_i \log(p_i)$.

a) $H(\hat{M}) = -\frac{1}{4}\log_2(\frac{1}{4}) - \frac{3}{4}\log_2(\frac{3}{4}) = \frac{1}{2} + \frac{3}{2} - \frac{3}{4}\log_2(3) \approx 0.811$
$H(\hat{K}) = -\frac{1}{2}\log_2(\frac{1}{2}) - 2\frac{1}{4}\log_2(\frac{1}{4}) = \frac{1}{2} + 1 = 1.5$

c	K_1	K_2	K_3	
a	1	2	3	$\frac{1}{4}$
b	2	3	4	$\frac{3}{4}$
	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	1

$$
\begin{aligned}
&P(\hat{C}=1) = P(\hat{M}=a)\cdot P(\hat{K}=K_1) = \tfrac{1}{4}\cdot\tfrac{1}{2} = \tfrac{1}{8}\\
&P(\hat{C}=2) = P(\hat{M}=a)\cdot P(\hat{K}=K_2) + P(\hat{M}=b)\cdot P(\hat{K}=K_1) = \tfrac{1}{4}\cdot\tfrac{1}{4} + \tfrac{3}{4}\cdot\tfrac{1}{2} = \tfrac{7}{16}\\
&P(\hat{C}=4) = P(\hat{M}=b)\cdot P(\hat{K}=K_3) = \tfrac{3}{4}\cdot\tfrac{1}{4} = \tfrac{3}{16}\\
\Rightarrow &P(\hat{C}=3) = 1 - P(\hat{C}=1) - P(\hat{C}=2) - P(\hat{C}=4) = 1 - \tfrac{2}{16} - \tfrac{7}{16} - \tfrac{3}{16} = \tfrac{4}{16}\\
\Rightarrow &H(\hat{C}) = -\tfrac{1}{8}\log_2(\tfrac{1}{8}) - \tfrac{7}{16}\log_2(\tfrac{7}{16}) - \tfrac{3}{16}\log_2(\tfrac{3}{16}) - \tfrac{1}{4}\log_2(\tfrac{1}{4}) \approx 1.850\\
\Rightarrow &H(\hat{K}\mid\hat{C}) \overset{\text{Thm. 4.7}}{=} H(\hat{M}) + H(\hat{K}) - H(\hat{C}) \approx 0.811 + 1.5 - 1.850 = 0.461
\end{aligned}
$$

b) Lem. 4.12 b) demands $|\mathcal{C}_+| \le |\mathcal{K}_+|$ for perfect secrecy.
But in this case, we get $4 = |\mathcal{C}_+| > |\mathcal{K}_+| = 3$ ↯

Solution of Problem 3

Show for any function $f : X(\Omega)\times Y(\Omega) \to \mathbb{R}$, that $H(X,Y,f(X,Y)) = H(X,Y)$.

By definition, we have:

$$H(X,Y,Z=f(X,Y)) \overset{\text{Def.}}{=} \sum_{X,Y,Z} P(X=x,Y=y,Z=z)\log\left(P(X=x,Y=y,Z=z)\right)$$

With

$$P(X=x,Y=y,Z=z) = \begin{cases} P(X=x,Y=y) & \text{, if } Z=f(X,Y)\\ 0 & \text{, if } Z\neq f(X,Y)\end{cases},$$

it follows that

$$H(X,Y,Z=f(X,Y)) = \sum_{X,Y} P(X=x,Y=y)\log(P(X=x,Y=y)) = H(X,Y)\,.$$

Note: It holds $0\cdot\log 0 = 0$.

Solution of Problem 4

a)

$$H(M) = -\sum_i P(M_i)log_2 P(M_i) = -(\frac{1}{3}log_2\frac{1}{3} + \frac{2}{3}log_2\frac{2}{3})$$

b) (i) For each $M \in \mathcal{M_N}, C \in \mathcal{C_N}$ there exists exactly one $K \in \mathcal{K_N}$ such that $e(M, K) = C$, namely $K = (s_1, \ldots, s_N)$ with $s_j = (c_j - a_j) \mod m$.

(ii) $\tilde{K}_N$ is uniformly distributed over $\mathcal{K_N}$, as

$$P(\tilde{K}_N = K) = P(\tilde{K}_1 = s_1, \ldots, \tilde{K}_N = s_N) = \prod_{i=1}^{N} P(\tilde{K}_i = s_i) = \frac{1}{m^N} = \frac{1}{|\mathcal{K_N}|}$$

$\forall K = (s_1, \ldots s_N)$

(iii) Disadvantage of Vernam Cipher: The main disadvantage of the Vernam Cipher is that : $|\mathcal{K}_+| \geq |\mathcal{M}_+|$ (one needs at least as many keys as plaintexts) and these keys need to be communicated over a secure channel in advance.

c)

$$H(C, M) \stackrel{chain-rule}{=} H(C) + H(M|C) \stackrel{perf.sec.}{=} H(C) + H(M)$$

d) Due to the independence of X and Y we have $p_Y(y|x) = p_Y(y)$, and

$$\tilde{H}(Y|X) = -\sum_x \sum_y p_Y(y)\, log_2\, p_Y(y) = |\mathcal{X}|H(Y) \geq H(Y)$$