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Tutorial 4

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Problem 1. (Definition of convexity) Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex $a, b \in \textbf{dom}\ f$ with $a < b$.

a) Show that

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

for all $x \in [a, b]$.

b) Show that

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$$

for all $x \in (a, b)$. Draw a sketch that illustrates this inequality.

c) Suppose that f is differentiable. Use the result in (b) to show that

$$f^{'}(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(b).$$

d) Suppose that f is twice differentiable. Use the result in (c) to show that $f^{''}(a) \geq 0$ and $f^{''}(b) \geq 0$.

Problem 2. (Second-order condition for convexity) Let $f : \mathcal{C} \to \mathbb{R}$ be a twice differentiable function on a convex set $\mathcal{C} \subset \mathbb{R}^n$. Prove the following statements.

a) Let $n = 1$, then f is convex, iff $f''(x) \geq 0$, $\forall x \in \mathcal{C}$.

b) f is convex, iff $\nabla^2 f(\boldsymbol{x}) \geq 0$, $\forall \boldsymbol{x} \in \mathcal{C}$.

Problem 3. (Inverse of an increasing convex function) Suppose $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and convex on its domain (a, b). Let g denote its inverse, i.e., the function with domain $(f(a), f(b))$ and $g(f(x)) = x$ for $a < x < b$. What can you say about convexity or concavity of g?