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Tutorial 14

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Problem 1.

a) Let $\mathcal{C} = \left\{ \boldsymbol{x} \in \mathbb{R}^2_{\geq 0} \mid x_1^2 x_2 + 2x_1 x_2 + x_2 \geq 1 \right\}$. Show that $\mathcal{C}$ is a convex set by rewriting it as an intersection of halfspaces.

b) Let $f(\boldsymbol{x}) = 1 + x_1 + 2x_2 + \frac{1}{2x_1 x_2}$, $\boldsymbol{x} \in \mathbb{R}^2_{>0}$. Decide whether f is a convex or concave function and give reasons.

c) Let $g : \mathbb{R}^n \mapsto \mathbb{R}$ be a differentiable and convex function with $g(\boldsymbol{x}) \leq b \in \mathbb{R}$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Show that g must be constant.

Problem 2.

A very important feature of Newton's method in unconstrained optimization is that it is affine invariant, i.e., it is independent of affine changes of coordinates. Consider a twice continuously differentiable and strictly convex function $f : \mathbb{R}^n \to \mathbb{R}$. The function $g : \mathbb{R}^n \to \mathbb{R}$ is defined as $g(\boldsymbol{x}) = f(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$ where $\boldsymbol{A} \in \mathbb{R}^{nxn}$ is non-singular and $\boldsymbol{b} \in \mathbb{R}^n$. In the following, we consider running Newton's method on $g(\boldsymbol{x})$ starting at some $\boldsymbol{x}_g^{(0)}$ and on $f(\boldsymbol{x})$ starting at $\boldsymbol{x}_f^{(0)} = \boldsymbol{A}\boldsymbol{x}_g^{(0)} + \boldsymbol{b}$.

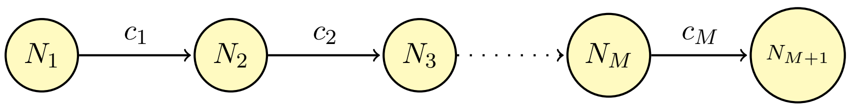
a) Let $\Delta\boldsymbol{x}_f$ and $\Delta\boldsymbol{x}_g$ be the Newton steps for $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$, respectively. Show that $\Delta\boldsymbol{x}_f^{(0)} = \boldsymbol{A}\Delta\boldsymbol{x}_g^{(0)}$.

b) Show that the step sizes t_f and t_g will be equal when exact line search is applied to $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$.

c) Use the results from part (a) and part (b) to prove that $\boldsymbol{x}_f^{(k)} = \boldsymbol{A}\boldsymbol{x}_g^{(k)} + \boldsymbol{b}$ and $f\left(\boldsymbol{x}_f^{(k)}\right) = g\left(\boldsymbol{x}_g^{(k)}\right)$ when exact line search is applied to $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$.

d) Prove that Newton's decrement λ_f^2 for $f(\boldsymbol{x})$ at $\boldsymbol{x}_f^{(k)}$ is equal to Newton's decrement λ_g^2 for $g(\boldsymbol{x})$ at $\boldsymbol{x}_g^{(k)}$, and so the stopping conditions are also identical.

Problem 3. In the multihop network consisting of $M+1$ nodes, see the figure below, the i^{th} link has capacity $c_i > 0$ and is operated for a fraction $x_i \in [0,1]$ of time where $\sum_{i=1}^{M} x_i = 1$. Thus, the effective rate on the i^{th} link is $x_i c_i$. Moreover, the end-to-end rate from node 1 to node $M+1$ is $\min_{i=1,\ldots,M}\{x_i c_i\}$, i.e., it is limited by the worst link.



a) Model the multihop network as an optimization problem with the objective to maximize the end-to-end rate from node 1 to node M+1 over the fraction of times $\boldsymbol{x}$.

b) Is this problem convex? Substantiate your answer.

c) Does it satisfy Slater's constraint qualification? Give a reason.

In the following, we consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & -t \\ \text{subject to} \quad & t - x_i c_i \leq 0, \quad i = 1, \ldots, M \\ & \sum_{i=1}^{M} x_i = 1 \\ & -x_i \leq 0 \end{aligned} \tag{1}$$

with optimization variables $\boldsymbol{x}$ and t.

d) Write the expression of the Langrangian for (1).

e) State the KKT conditions.

f) Solve the KKT conditions to get the solution

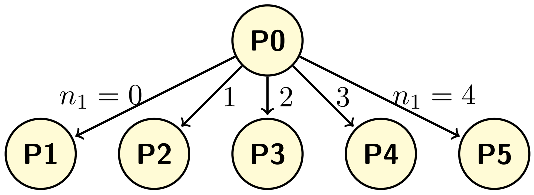
$$x_j = \frac{1/c_j}{\sum_{i=1}^{M} c_i}, \quad j = 1, \ldots, M.$$

Problem 4. An institute has been granted a budget of maxK $= 7$ for computers. The institute has space for maxC $= 4$ computers. At least minG $= 3$ high performance graphics cards shall be included. From the seller the institue gets the information that there are three different types available. They all have different performances $\boldsymbol{l} = (l_1, l_2, l_3)$, different number of high performance graphics cards per computer $\boldsymbol{g} = (g_1, g_2, g_3)$ and different costs $\boldsymbol{k} = (k_1, k_2, k_3)$. Let $\boldsymbol{n} = (n_1, n_2, n_3)$ be the notation for the number of computers which are bought. Help the institute in making the optimal choice in terms of maximizing the performance while considering the above mentioned constraints by answering the following tasks.

a) Formulate the optimization problem with all constraints.

b) What kind of optimization problem is it?

The performances are given as $\boldsymbol{l} = (1, 2, 3)$, the number of graphics cards per computer as $\boldsymbol{g} = (2, 1, 0)$ and the costs as $\boldsymbol{k} = (1, 3, 2)$. A solver returned for the relaxed problem (nonnegative real values are allowed for the number of computers) the solution $(4/3, 1/3, 7/3)$ and objective value of 9. Unfortunately, this is not an integer solution, therefore, the branch-and-bound method is applied as depicted in the following graph.



c) The solver returns $(2, 0, 2)$ at node P3 for the relaxed problem. What objective value is attained? Do you need to process this subtree further?

Assume in the following that the current best objective value is 8.

d) Evaluate the relaxed problem at node P2 by tackling the following tasks.

i) Formulate the optimization problem to be solved in P2.

ii) Solve the optimization problem graphically.

iii) Calculate the optimal point and value.

iv) Is it necessary to process node P2 further? Give a reason.