Widely-Linear Processing for Distributed Passive Radar Systems with Strictly Non-Circular Sources

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Abstract—**In this paper, we address the design of a distributed passive radar system which operates on a strictly non-circular source. The goal of a passive distributed radar is to provide a reliable estimation from the source signal, by collecting and combining the individual observations from the network in a centralized node. We take advantage of the non-circular nature of the source, and propose widely linear signal processing algorithms, which achieve around 3 dB of gain in the estimation accuracy compared to the available linear strategies. In this regard, a minimum mean squared error (MMSE) problem is formulated for unbiased class of estimators, where the widely-linear processing is enabled at the distributed sensors, or at a centralized entity. An optimal power allocation and information fusion is achieved in each case, by studying the optimality conditions of the corresponding problems. The performance of the proposed methods are then compared to the known linear strategies via numerical simulations.**

I. Introduction

In the recent years the applications of radar sensor networks have gained significance, due to the high operational diversity and scalability. Such use cases have been successfully presented, e.g., in [1]–[5] for bi- and multi-static radars. In particular, applications of a system of passive distributed radar have been exemplified in [2] in the context of remote surveillance, in [5] addressing a high resolution 3-D imaging problem. The goal of a distributed passive radar is to provide a reliable estimation from a source signal, by collecting and combining the individual passive observations from a network of sensor nodes (SN)s in a centralized node, i.e., fusion center (FC). As an interesting example, we can mention the '*IceCube Neutrino Observatory*' at the south pole, where a sensor network with more than 5000 sensor nodes (SNs) is deployed to observe certain characteristics of sub-atomic particles [4]. Since the operation of the whole sensor network is mostly intended to consume minimum resources while keeping the individual cost and maintenance of SNs low, an energy efficient operation is highly desirable. Hence, the related problem of optimal power allocation and corresponding energy-aware system design has been addressed in many works, e.g., in [6], [7] and [8]. In

These studies are then extended to include considerations regarding computational complexity [9], network lifetime and energy efficiency [10], [11], and occasional node failure [12].

The aforementioned works focus on the linear signal processing strategies at the SNs and at the FC, motivated by the computation simplicity, as well as the closed form nature of the resulting optimal solutions. On the other hand, for a communication or a radar system with a non-circular signal source, where the real and imaginary parts of the signal are correlated, the application of a widely-linear (WL) processing strategy is promising[1]. Popular examples of such sources are the ones following a binary phase shift keying (BPSK), amplitude shift keying (ASK) or an offset quadrature amplitude modulation (OQAM) signal constellations, see [14] for more examples. In [15] it is shown that the achievable end-to-end communication rate can be significantly enhanced for a synchronized relay network with a non-circular source, via the application of WL beamforming. The work in [14] presents various use cases for exploiting the source non-circularity in the context of radar signal processing. In particular, for a system of passive distributed radar, the application of WL processing at multiple network nodes is expected to be gainful. This is since the non-circular distribution of the source signal can be exploited, via a WL process, to better distinguish the desired signal from the noise components which follow a circularly-symmetric distribution. Nevertheless, the available system-level designs have not yet been extended to realize this potential.

Contribution: In this work, we extend the proposed linear design strategies in [7], addressing the distributed passive radar applications, for the case that the source signal follows a strictly non-circular distribution. In this regard, a minimum mean squared error (MMSE) problem is formulated for unbiased class of estimators, where the widely-linear processing is enabled at the SNs, or at the FC. An optimal power allocation and information fusion is achieved in each case, by studying the optimality conditions of the corresponding problems. The performance of the proposed methods are then compared to the optimal linear strategies via numerical simulations.

II. System Model

We investigate a network of K passive SNs, cooperating to achieve a single global observation via a FC. Both communication and sensing channels are wireless, and follow the quasi-static[2] frequency-flat fading model. The final goal of each observation is to classify (or detect) a strictly non-circular source signal r at the FC. Each observation process can be segmented into three parts: sensing process, communication process and information fusion. For a detailed similar network description with a circular source and linear process, please see [7, Section II].

[1] A WL process may use conjugation (which is not a linear operator) in addition to the linear operators. This enables a WL process to separate the real and imaginary signal parts and exploit the correlation among them in non-circular signals, see [13] for more details.

[2] It represents that the channel is constant during each observation, but may vary from observation to another.

A. Strict Non-Circularity at the Source

In this work we focus on a setup with a zero-mean and strictly non-circular source, $r \in \mathbb{C}$. In a complex-valued signal with a circularly symmetric distribution, the imaginary and real parts of the signal are independent. On the other hand, for a strictly non-circular source, the signal constellation is located on a straight line with a constant phase, i.e., $r = |r|e^{j\theta_0}$ where θ_0 is constant for all constellation points, see [16, Equation 2]. The statistics of the source can be hence expressed as

$$R := \mathcal{E}\{|r|^2\} = |\mathcal{E}\{r^2\}|, \;\; \mathcal{E}\{r\} = 0, \tag{1}$$

where R is the source signal power and $\mathcal{E}\{\cdot\}$ represents mathematical expectation. Note that the above argument describes the opposite situation compared to a circular source where the constellation points are circular-symmetrically distributed and hence we have $\mathcal{E}\{r^2\} = 0$. In this work, without loss of generality, we focus on real-valued source constellation, $r \in \mathbb{R}$[1]. For further elaboration on the statistical properties of non-circular signals please see [16].

B. Operation of SNs

If a target signal r is present, it will be received at each SN together with an additive white zero-mean complex Gaussian-distributed circularly-symmetric (AWZMCGCS) noise. The received signal is then processed at each SN and transmitted to FC. The communication with FC is performed by using orthogonal waveforms for each SN so that data from different SNs can be separated and processed in FC. The process of each SN can be described as

$$y_k := g_k r + m_k, \;\; x_k := \mathbf{f}_k^{(\text{SN})}(y_k) \tag{2}$$

and

$$X_k := \mathcal{E}\{|x_k|^2\}, \tag{3}$$

where $\mathbf{f}_k^{\text{SN}}(\cdot)$ represents the signal processing at the SN with index k. The sensing channel coefficient, communication signal and its power from the SN with index k is respectively denoted by $g_k \in \mathbb{C}$, $x_k \in \mathbb{C}$ and X_k. The AWZMCGCS noise on the sensing process and its variance are respectively denoted as $m_k \in \mathbb{C}$ and M_k. In order to take into account the weak energy storage capability of the SNs, we consider a maximum allowed individual average power consumption on each SN. Furthermore, we limit the total network power consumption to adjust the required network life-time to the available power budget. The aforementioned constraints are expressed as

$$X_k \le P_k \Leftrightarrow \mathcal{E}\{\left|\mathbf{f}_k^{(\text{SN})}(y_k)\right|^2\} \le P_k, \tag{4}$$

and

$$\sum_{k \in \mathbb{F}_K} X_k \le P_{\text{tot}} \Leftrightarrow \sum_{k \in \mathbb{F}_K} \mathcal{E}\{\left|\mathbf{f}_k^{(\text{SN})}(y_k)\right|^2\} \le P_{\text{tot}}, \tag{5}$$

where $P_k \in \mathbb{R}_+$ and $P_{\text{tot}} \in \mathbb{R}_+$ respectively represent the individual and total power constraints on the function of SNs.

[1]This follows as the constant term $e^{j\theta_0}$ can be considered as a part of the channel between source and SNs, as it remains constant for different observations and constellation points.

C. Fusion Center

The transmitted signal from each SN passes through the communication channel, with coefficients $h_k \in \mathbb{C}$, and arrives at the FC combined with an AWZMCGCS noise component n_k, with variance N_k. A signal processing stage is then applied at the FC to achieve an estimation, $\tilde{r}$, of the source signal from the collected observations in the network. This is described as

$$z_k := h_k x_k + n_k, \tag{6}$$

and

$$\tilde{r} := \mathbf{f}^{(\text{FC})}(z_1, \cdots, z_K), \tag{7}$$

where $\mathbf{f}^{\text{FC}}(\cdot)$ represents information fusion strategy, i.e., the signal processing at the FC.

D. Remarks

- In the present work we assume the availability of the perfect channel information for both sensing and communication channels. In general, it is rather difficult to estimate the sensing channel in an accurate way unless the channel has a highly stationary nature, e.g., [4]. Hence, for scenarios where the sensing channel is not stationary, the results of this paper can be treated as theoretical limits.
- In this work we focus on the systems with a strictly non-circular source. Nevertheless, the proposed solutions can remain gainful for a reduced degree of non-circularity at the source. The sensitivity of the network performance to the non-circularity of the source is numerically studied in Section V.
- In the present work we apply WL processing at the SNs, or at the FC. The joint application of WL processing at both SNs and the FC will be the goal of our future investigations.

III. Optimal Widely-Linear Fusion with Linear Signal Processing at the SNs

In this part we investigate the scenario where FC applies a WL process on the received signals from the SNs, while each SN performs a linear process, similar to the SN operation in [7]. This is described as

$$\mathbf{f}_k^{(\text{SN})}(a) := u_k a, \;\; \forall k \in \mathbb{F}_K \;\; \Rightarrow \\ x_k = u_k\left(g_k r + m_k\right), \;\; \forall k \in \mathbb{F}_K, \tag{8}$$

and

$$\mathbf{f}^{(\text{FC})}(a_1, \cdots, a_K) := \sum_{k \in \mathbb{F}_K} \hat{v}_k a_k + \check{v}_k a_k^* \;\; \forall k \in \mathbb{F}_K \;\; \Rightarrow \\ \tilde{r} = \sum_{k \in \mathbb{F}_K} \hat{v}_k z_k + \check{v}_k z_k^* \;\; \forall k \in \mathbb{F}_K, \tag{9}$$

where $u_k \in \mathbb{R}$ represents the linear amplification coefficient at the k-th SN and $\hat{v}_k, \check{v}_k \in \mathbb{C}$ are the linear fusion weights which apply a widely linear process on z_k at the FC [3]. Via the

[3]Please note that the real-valued assumption of the amplification coefficients does not reduce the generality, since any amplification phase can be compensated via a constant rotation of fusion weights, with no effect on the system objective and constraints. Please refer to [12, Lemma 1] for a similar discussion.

application of (8), the defined power constraints in (4) and (5) are formulated as

$$X_k \leq P_k \Leftrightarrow |u_k|^2 \left(R|g_k|^2 + M_k\right) \leq P_k, \tag{10}$$

and

$$\sum_{k\in\mathbb{F}_K} X_k \leq P_{\text{tot}} \Leftrightarrow \sum_{k\in\mathbb{F}_K} |u_k|^2 \left(R|g_k|^2 + M_k\right) \leq P_{\text{tot}}. \tag{11}$$

Furthermore, via the application of (9) into (7) the final estimation at the FC is formulated as

$$\begin{aligned}\tilde{r} = r &\sum_{k\in\mathbb{F}_K} \hat{v}_k u_k h_k g_k + \check{v}_k u_k h_k^* g_k^* \\ &+ \sum_{k\in\mathbb{F}_K} \hat{v}_k u_k h_k m_k + \check{v}_k u_k h_k^* m_k^* \\ &+ \sum_{k\in\mathbb{F}_K} \hat{v}_k n_k + \check{v}_k n_k^*,\end{aligned} \tag{12}$$

where (12) follows as the identity $r = r^*$ holds for $r \in \mathbb{R}$. Following the previous designs with a linear fusion process [6], [10]–[12], we focus on unbiased class of estimators and choose mean-squared-error (MSE) as our estimation metric. The unbiased estimation condition can be expressed as

$$\begin{aligned}&\mathcal{E}\{\tilde{r} - r\} = 0 \quad \Leftrightarrow \\ &\sum_{k\in\mathbb{F}_K} \hat{v}_k u_k h_k g_k + \check{v}_k u_k h_k^* g_k^* = 1,\end{aligned} \tag{13}$$

where (13) follows as the noise terms are all zero-mean. By applying (13) into (12) and considering the fact that all of the noise terms are zero-mean, circularly symmetric, and mutually independent, the estimation MSE is obtained as

$$\begin{aligned}V :&= \mathcal{E}\{|\tilde{r} - r|^2\} \\ &= \sum_{k\in\mathbb{F}_K} |u_k|^2 \left(|\hat{v}_k|^2 + |\check{v}_k|^2\right) |h_k|^2 M_k \\ &+ \sum_{k\in\mathbb{F}_K} \left(|\hat{v}_k|^2 + |\check{v}_k|^2\right) N_k.\end{aligned} \tag{14}$$

Our optimization strategy in order to obtain an unbiased minimum MSE (MMSE) estimation is hence expressed as

$$\min_{u_k\in\mathbb{R}, \hat{v}_k, \check{v}_k,\ k\in\mathbb{F}_K} \quad V \quad \text{s.t. (13), (10), (11),} \tag{15}$$

where the constraints in (15) result in a system parameter set that satisfy the unbiased estimation, while the defined power constraints are satisfied. As it can be seen from (14), the resulting problem is not a jointly convex problem over the optimization variables. Nevertheless, it is a separately convex optimization problem over the amplification coefficients, i.e., u_k, and over the fusion weights, i.e., $\hat{v}_k, \check{v}_k$, $\forall k \in \mathbb{F}_K$. Hence, in the next step we obtain the optimal fusion rule for a given (fixed) set of amplification coefficients.

A. Optimal WL Fusion

As observed from (4) and (5), the power constraints are invariant to the choice of the fusion weights. For a fixed set of u_k, $\forall k \in \mathbb{F}_K$ that satisfies the defined power constraints, the optimization over the fusion weights can be formulated as

$$\min_{\hat{v}_k, \check{v}_k\ k\in\mathbb{F}_K} \quad V \quad \text{s.t. (13),} \tag{16}$$

which holds a convex structure. The corresponding Lagrangian function is consequently formulated as

$$\begin{aligned}\mathcal{L}\left(\hat{v}_k, \check{v}_k, \lambda\right) := \lambda &\left(1 - \sum_{k\in\mathbb{F}_K} \hat{v}_k u_k h_k g_k + \check{v}_k u_k h_k^* g_k^*\right) \\ &+ \sum_{k\in\mathbb{F}_K} u_k^2 \left(|\hat{v}_k|^2 + |\check{v}_k|^2\right) |h_k|^2 M_k \\ &+ \sum_{k\in\mathbb{F}_K} \left(|\hat{v}_k|^2 + |\check{v}_k|^2\right) N_k.\end{aligned} \tag{17}$$

where $\mathcal{L}(\cdot)$ represents the Lagrangian, and λ is the dual variable corresponding to the unbiased estimation constraint. Due to the convex problem structure, any optimum solution to (16) will be located at the stationary point of the defined Lagrangian function. Following the guidelines of the Wirtinger calculus [17], [18], the derivative of the Lagrangian function can be obtained with respect to $\hat{v}_k$ and $\check{v}_k$ as

$$\frac{\partial \mathcal{L}}{\partial \hat{v}_k} = \hat{v}_k^* \left(u_k^2 |h_k|^2 M_k + N_k\right) - \lambda u_k h_k g_k = 0, \ \ \forall k \in \mathbb{F}_K, \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial \check{v}_k} = \check{v}_k^* \left(u_k^2 |h_k|^2 M_k + N_k\right) - \lambda u_k h_k^* g_k^* = 0, \ \ \forall k \in \mathbb{F}_K. \tag{19}$$

Consequently we can write

$$\sum_{k\in\mathbb{F}_K} \hat{v}_k \frac{\partial \mathcal{L}}{\partial \hat{v}_k} + \check{v}_k \frac{\partial \mathcal{L}}{\partial \check{v}_k} = 0 \quad \Rightarrow \quad V = \lambda. \tag{20}$$

Furthermore, applying the identity

$$\sum_{k\in\mathbb{F}_K} \frac{h_k^* g_k^* u_k}{u_k^2 |h_k|^2 M_k + N_k} \times \frac{\partial \mathcal{L}}{\partial \hat{v}_k} + \frac{h_k g_k u_k}{u_k^2 |h_k|^2 M_k + N_k} \times \frac{\partial \mathcal{L}}{\partial \check{v}_k} = 0, \tag{21}$$

we have

$$V = \frac{1}{2} \left(\sum_{k\in\mathbb{F}_K} \frac{|h_k g_k|^2 u_k^2}{u_k^2 |h_k^2| M_k + N_k}\right)^{-1}. \tag{22}$$

By plugging (22) and (20) into (18) and (19) we can calculate

$$\hat{v_k}^\star = \left(\sum_{k\in\mathbb{F}_K} \frac{|h_k g_k|^2 u_k^2}{u_k^2 |h_k^2| M_k + N_k}\right)^{-1} \frac{\frac{1}{2} h_k^* g_k^* u_k}{u_k^2 |h_k^2| M_k + N_k}, \tag{23}$$

$$\begin{aligned}\check{v_k}^\star &= \left(\sum_{k\in\mathbb{F}_K} \frac{|h_k g_k|^2 u_k^2}{u_k^2 |h_k^2| M_k + N_k}\right)^{-1} \frac{\frac{1}{2} h_k g_k u_k}{u_k^2 |h_k^2| M_k + N_k} \\ &= \left(\hat{v_k}^\star\right)^*\end{aligned} \tag{24}$$

where $\hat{v_k}^\star$ and $\check{v_k}^\star$ respectively represent the optimal fusion coefficients $\hat{v_k}$ and $\check{v_k}$.

B. Optimal Linear Processing at the SNs: Power Allocation Problem

Via the application of the obtained fusion weights, i.e, (23), (24), we aim at minimizing the resulting MSE, see (22), while satisfying the defined power constraints (10), (11). This can be equivalently formulated as

$$\min_{u_k \in \mathbb{R}} \quad - \sum_{k \in \mathbb{F}_K} \frac{|h_k g_k|^2 u_k^2}{u_k^2 |h_k^2| M_k + N_k} \qquad \text{s.t. } (10), (11), \tag{25}$$

which is a convex optimization problem, and results in a water-filling solution structure. For an analytical optimum solution for a similar problem please refer to [12, Section III], [7].

C. Interpretation of the Solution

As it can be observed from (22), in comparison with [7, Equation (21)], benefiting from the strictly non-circular nature of the source signal, the resulting MSE will be reduced by factor of two for an optimal WL process at the FC compared to the optimal linear processing. Furthermore, the values of the WL fusion weights are complex conjugate of each other. In the other words, the desired signal components (source signal) are aligned at the FC on the real axis, via the application of (23) and (24). In this way, the remaining imaginary parts are safely ignored as they result from the additive noise at the FC or at the SNs.

IV. Optimal Widely-Linear Processing at the SNs with Linear Processing at the FC

In this part we investigate a scenario where the FC applies a linear process on the received signals from the SNs, while each SN performs a widely linear process on the received signal from the sensing channel. This is expressed as

$$\begin{aligned} \mathbf{f}_k^{(\mathrm{SN})}(a) &:= \hat{u}_k a + \check{u}_k a^*, \quad \forall k \in \mathbb{F}_K \quad \Rightarrow \\ x_k &= \hat{u}_k (g_k r + m_k) + \check{u}_k (g_k r + m_k)^*, \quad \forall k \in \mathbb{F}_K, \end{aligned} \tag{26}$$

and

$$\begin{aligned} \mathbf{f}^{(\mathrm{FC})}(a_1, \cdots, a_K) &:= \sum_{k \in \mathbb{F}_K} v_k a_k \quad \forall k \in \mathbb{F}_K \quad \Rightarrow \\ \tilde{r} &= \sum_{k \in \mathbb{F}_K} v_k z_k \quad \forall k \in \mathbb{F}_K, \end{aligned} \tag{27}$$

where $v_k \in \mathbb{C}$ is the linear fusion weight, and $\hat{u}_k, \check{u}_k \in \mathbb{C}$ are the amplification coefficients at the SN, respectively applied to the actual and the complex-conjugate versions of the received signal. Via the application of (26), the defined power constraints in (4) and (5) are formulated as

$$\begin{aligned} & X_k \leq P_k \Leftrightarrow \\ & M_k \left(|\hat{u}_k|^2 + |\check{u}_k|^2 \right) + R |\hat{u}_k g_k + \check{u}_k g_k^*|^2 \leq P_k, \end{aligned} \tag{28}$$

and

$$\begin{aligned} & \sum_{k \in \mathbb{F}_K} X_k \leq P_{\text{tot}} \Leftrightarrow \\ & \sum_{k \in \mathbb{F}_K} M_k \left(|\hat{u}_k|^2 + |\check{u}_k|^2 \right) + R |\hat{u}_k g_k + \check{u}_k g_k^*|^2 \leq P_{\text{tot}}. \end{aligned} \tag{29}$$

Furthermore, via the application of (26) and (6) into (27) the final estimation at the FC is formulated as

$$\begin{aligned} \tilde{r} = r & \sum_{k \in \mathbb{F}_K} v_k h_k \left(\hat{u}_k g_k + \check{u}_k g_k^* \right) \\ & + \sum_{k \in \mathbb{F}_K} v_k h_k \left(\hat{u}_k m_k + \check{u}_k m_k^* \right) + \sum_{k \in \mathbb{F}_K} v_k n_k, \end{aligned} \tag{30}$$

where (30) follows as the identity $r = r^*$ holds for $r \in \mathbb{R}$. Similar to (13), the unbiased estimation constraint is expressed as

$$\begin{aligned} & \mathcal{E}\{\tilde{r} - r\} = 0 \quad \Leftrightarrow \\ & \sum_{k \in \mathbb{F}_K} v_k h_k \left(\hat{u}_k g_k + \check{u}_k g_k^* \right) = 1. \end{aligned} \tag{31}$$

By applying (31) into (30) and considering the fact that all of the noise terms are zero-mean, circularly symmetric, and mutually independent, the estimation MSE is obtained as

$$\begin{aligned} V :&= \mathcal{E}\{|\tilde{r} - r|^2\} \\ &= \sum_{k \in \mathbb{F}_K} |v_k|^2 |h_k|^2 M_k \left(|\hat{u}_k|^2 + |\check{u}_k|^2 \right) + \sum_{k \in \mathbb{F}_K} |v_k|^2 N_k. \end{aligned} \tag{32}$$

Hence for the scenario with a WL process at the SNs and linear process at the FC we can express our optimization problem as

$$\min_{v_k, \hat{u}_k, \check{u}_k \; k \in \mathbb{F}_K} \quad V \quad \text{s.t. } (31), (28), (29), \tag{33}$$

which represents an MMSE optimization within the unbiased class of estimators, while satisfying the defined power constraints (28) and (29). The resulting problem is not a jointly convex problem over the optimization variables. Nevertheless, it is a separately convex optimization problem over the amplification coefficients, i.e., $\hat{u}_k$ and $\check{u}_k$, and over the fusion weights, i.e., v_k, $\forall k \in \mathbb{F}_K$. Hence, in the next step we obtain the optimal fusion rule for a given (fixed) set of amplification coefficients.

A. Optimal Linear Fusion

Similar to Subsection III-A, the power constraints are invariant to the choice of the fusion weights, see (28) and (29). For a fixed set of $\hat{u}_k, \check{u}_k$, $\forall k \in \mathbb{F}_K$, that satisfies the defined power constraints, the optimization over the fusion weights can be formulated as

$$\min_{v_k, \; \forall k \in \mathbb{F}_K} \quad V \quad \text{s.t. } (31), \tag{34}$$

which holds a convex structure. The following lemma reveals the phase characteristics of the fusion weights v_k at the optimality.

Lemma 1: Let $v_1, \ldots, v_K$ be a feasible set of fusion coefficients. Then the following variable update:

$$\bar{v}_k \leftarrow |v_k| \frac{h_k^* \left(\hat{u}_k g_k + \check{u}_k g_k^* \right)^*}{|h_k \left(\hat{u}_k g_k + \check{u}_k g_k^* \right)| \sum_{k \in \mathbb{F}_K} |v_k h_k \left(\hat{u}_k g_k + \check{u}_k g_k^* \right)|}, \tag{35}$$

where $\bar{v}_1, \ldots, \bar{v}_K$ represent a new set of fusion coefficients, is feasible and does not degrade the estimation, i.e., does not increase the objective function.

Proof: Please see Appendix.A. ■

Note that a clear interpretation of Lemma 1 is the fact that the received signals at the FC should be constructively aligned in order to reduce the effect of noise. As a useful corollary of Lemma 1 we can assume $\angle v_k = -\angle h_k \left(\hat{u}_k g_k + \check{u}_k g_k^* \right)$, without reducing the optimality. The aforementioned assumption simplifies our problem into finding $|v_k| \in \mathbb{R}^+, \forall k \in \mathbb{F}_K$. The corresponding Lagrangian function is hence formulated as

$$\mathcal{L}(|v_k|, \lambda) := \lambda \left(1 - \sum_{k\in\mathbb{F}_K} |v_k||h_k(\hat{u}_k g_k + \check{u}_k g_k^*)|\right) + \sum_{k\in\mathbb{F}_K} |v_k|^2|h_k|^2 M_k\left(|\hat{u}_k|^2 + |\check{u}_k|^2\right) + \sum_{k\in\mathbb{F}_K} |v_k|^2 N_k, \quad (36)$$

where $\mathcal{L}(\cdot)$ represents the Lagrangian, and λ is the dual variable corresponding to the unbiased estimation constraint. At every stationary point of the defined Lagrangian the derivatives will vanish with respect to $|v_k|$:

$$\frac{\partial \mathcal{L}}{\partial |v_k|} = 2|v_k|\left(M_k|h_k|^2\left(|\hat{u}_k|^2 + |\check{u}_k|^2\right) + N_k\right) - \lambda|h_k(\hat{u}_k g_k + \check{u}_k g_k^*)| = 0, \ \ \forall k \in \mathbb{F}_K. \quad (37)$$

Consequently we can write

$$\sum_{k\in\mathbb{F}_K} |v_k| \frac{\partial \mathcal{L}}{\partial |v_k|} = 0 \ \ \Rightarrow \ \ V = \frac{\lambda}{2}. \quad (38)$$

Furthermore, following a similar derivation as in (21) we have

$$\sum_{k\in\mathbb{F}_K} \frac{\partial \mathcal{L}}{\partial |v_k|} \times \frac{|h_k(\hat{u}_k g_k + \check{u}_k g_k^*)|}{\left(|\hat{u}_k|^2 + |\check{u}_k|^2\right)|h_k|^2 M_k + N_k} = 0, \quad (39)$$

and consequently

$$V = \left(\sum_{k\in\mathbb{F}_K} \frac{|h_k|^2|\hat{u}_k g_k + \check{u}_k g_k^*|^2}{\left(|\hat{u}_k|^2 + |\check{u}_k|^2\right)|h_k^2| M_k + N_k}\right)^{-1}. \quad (40)$$

By plugging (40) and (38) into (37), and following the result of Lemma 1 we calculate

$$v_k{}^\star = \left(\sum_{k\in\mathbb{F}_K} \frac{|h_k|^2|\hat{u}_k g_k + \check{u}_k g_k^*|^2}{\left(|\hat{u}_k|^2 + |\check{u}_k|^2\right)|h_k^2| M_k + N_k}\right)^{-1} \times \frac{h_k^*\left(\hat{u}_k g_k + \check{u}_k g_k^*\right)^*}{\left(|\hat{u}_k|^2 + |\check{u}_k|^2\right)|h_k|^2 M_k + N_k}, \quad (41)$$

where $v_k{}^\star, \forall k \in \mathbb{F}_K$ represent an optimal set of fusion coefficients.

B. Optimal Widely-Linear Processing at the SNs: Power Allocation Problem

In Subsection IV-A the optimal linear fusion rule is studied for a given set of amplification coefficients $\hat{u}_k, \check{u}_k, \forall k \in \mathbb{F}_K$. It is observed that an optimal unbiased linear fusion results in the estimation MSE equal to (40). In this part, we look for an optimal set of amplification coefficients which lead to the minimum V, see (40), while satisfying the defined power constraints (28), (29). The corresponding optimization problem can be hence formulated as

$$\max_{\hat{u}_k, \check{u}_k,\ k\in\mathbb{F}_K} \sum_{k\in\mathbb{F}_K} \frac{|h_k|^2|\hat{u}_k g_k + \check{u}_k g_k^*|^2}{\left(|\hat{u}_k|^2 + |\check{u}_k|^2\right)|h_k^2| M_k + N_k} \quad \text{s.t.} \quad (28),\ (29). \quad (42)$$

The following lemma reveals an important property of the amplification coefficients at the optimality.

Lemma 2: There exists an optimal solution to (42) where we have

$$\hat{u}_k^\star = \check{u}_k^{\star *}, \ \angle\hat{u}_k^\star = -\angle g_k. \quad (43)$$

Proof: Please see Appendix.B. ■

By benefiting from the results of the Lemma 2, the simplified version of our optimization problem in (42) is expressed as

$$\max_{\tilde{u}_k\in\mathbb{R}^+, k\in\mathbb{F}_K} \sum_{k\in\mathbb{F}_K} \frac{4\tilde{u}_k|h_k g_k|^2}{2\tilde{u}_k|h_k^2|M_k + N_k} \quad \text{s.t.} \quad \tilde{u}_k M_k + 2R\tilde{u}_k|g_k|^2 \le \frac{P_k}{2}, \quad \sum_{k\in\mathbb{F}_K} \tilde{u}_k M_k + 2R\tilde{u}_k|g_k|^2 \le \frac{P_{\text{tot}}}{2} \quad (44)$$

where $\tilde{u}_k := |\hat{u}_k|^2 = |\check{u}_k|^2$, and the optimal amplification coefficients can be obtained via Lemma 2 as

$$\hat{u}_k^\star = \frac{g_k^*\sqrt{\tilde{u}_k^\star}}{|g_k|}, \ \check{u}_k^\star = \frac{g_k\sqrt{\tilde{u}_k^\star}}{|g_k|}, \quad (45)$$

where $\tilde{u}_k^\star$ represents the optimal solution to (44). The defined problem (44) is a convex optimization problem and results in a water-filling solution structure. For an analytical optimum solution for a similar problem please refer to [12, Section III], [7].

C. Interpretation of the Solution

Similar to the application of WL processing at the FC, the defined WL process at the SNs aligns the components of the source signal on the real axis and eliminates the imaginary parts of the signal, caused by the noise. Nevertheless, the noise components at the FC can not be reduced with this strategy, as the WL process is merely applied at the SNs. As a result the defined WL processing at the SNs is less effective compared to the proposed WL processing at the FC.

V. Simulation Results

In this part we investigate the performance of the defined system via numerical simulations. We simulate a network with $K = 300$ SNs, where all sensing and communication channels are zero-mean and follow a Gaussian distribution with variance σ_g^2 and σ_h^2, respectively. For each set of channel realizations, i.e, $h_k, g_k, \forall k \in \mathbb{F}_K$, 10000 realizations of $r, n_k, m_k, \ \forall k \in \mathbb{F}_K$ are generated, following the defined statistics. The resulting network performance is then averaged over 100 channel realizations. We compare the proposed WL processing strategies in Section III and Section IV to the available linear strategies in terms of R/V. This includes the WL or linear design strategy at the SNs or at the FC. Regarding the power allocation strategies, our comparison includes the optimal power allocation as defined in Subsection III.B and Subsection IV.B, the equal power allocation (EPA) among all SNs and the allocation of the all available power to an optimally-selected single SN (SSN), see [19, Subsection III.C-E]. Table 1 defines the different design strategies which are evaluated in Fig. 2-4. Unless stated otherwise, the given values in Table 2 are used as the simulated network parameters.

TABLE I: Simulated design strategoies

Legend	Description
WL-FC-Opt	Optimal WL proc. at FC and linear proc. at SNs,
WL-SNs-Opt	Optimal WL proc.at SNs and linear proc. at FC
L-Opt	Optimal linear proc. at SNs and FC
WL-FC-EPA	Optimal WL proc.at FC with equal power alloc. among SNs
WL-SNs-EPA	WL proc. at SNs with equal power allocation among SNs
L-EPA	Optimal linear proc. at FC with equal power alloc. among SNs
WL-FC-SSN	WL proc. at FC with best sensor selection
WL-SNs-SSN	WL proc. at SN with best sensor selection
L-SSN	Linear proc. at SN and FC with best sensor selection

TABLE II: Reference simulation parameters

K	R	M_k	N_k	σ_g^2	σ_h^2	P_k	P_{tot}
300	1	1	1	1	1	1	60

In Fig. 1 the resulting estimation accuracy is depicted for different noise levels. It is apparent that higher noise variance results in the lower estimation quality. Furthermore, for various power allocation schemes, the application of WL processing at FC results in 3-dB of gain in the estimation accuracy, for various noise conditions. Relatively smaller gain is observed for the scenario with WL processing at the SNs. This is since the circular nature of the noise at the FC is not exploited in the latter case.

In Fig. 2 the impact of the source signal power is observed on the resulting estimation accuracy. As it is expected, higher source signal power results in a higher estimation accuracy. Similar performance gain margins are observed as to Fig. 1 for the application of WL processing at the FC and SNs, for various power allocation strategies.

While the proposed designs in Section III and IV are optimal only for a strictly non-circular source situation, they are still gainful in the presence of source circularity mismatch. Nevertheless, the level of tolerable mismatch is highly dependent on the noise variance. In Fig. 3 the performance of the proposed WL methods are compared to the available linear strategies, for various levels of source non-circularity coefficient[4]. Furthermore, it is observed that for a high variance of the noise signal, the proposed designs are gainful for a wider range of non-circularity coefficients.

VI. Conclusion

In this work we have developed WL signal processing methods for a passive distributed radar system, where the source signal follows a strictly non-circular distribution. We have observed that for an optimal WL processing at the FC, we can obtain 3 dB of gain in the resulting estimation accuracy, compared to the available optimal linear strategies. Nevertheless, smaller gain margin was observed for an optimal WL processing at the SNs. While the proposed WL methods rely on the strict non-circularity of the source signal, as it is numerically observed, they remain gainful for weak non-circularity condition at the source depending on the network noise situation.

[4]This represents the level of non-circularity for the source signal distribution, i.e., $\rho := \frac{\mathcal{E}\{r^2\}}{\mathcal{E}\{|r|^2\}}$. $\rho = 1$ represents a strictly non-circular distribution, while $\rho = 0$ represents a circular distribution, please also see [13] for more details.

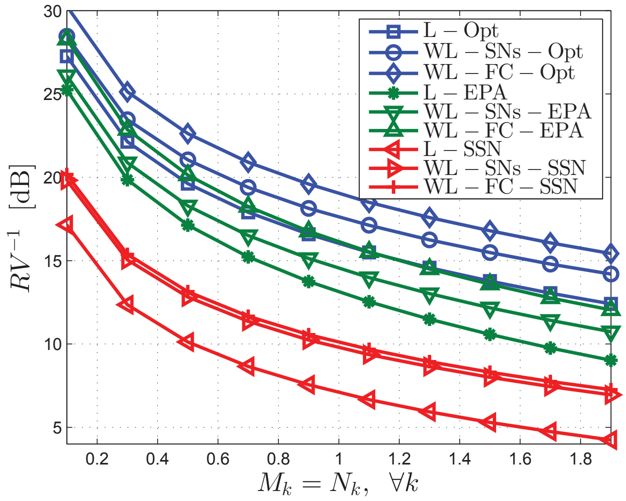

Fig. 1: Estimation accuracy in terms of RV^{-1} [dB] vs. Noise power $M_k = N_k$ [Watt].

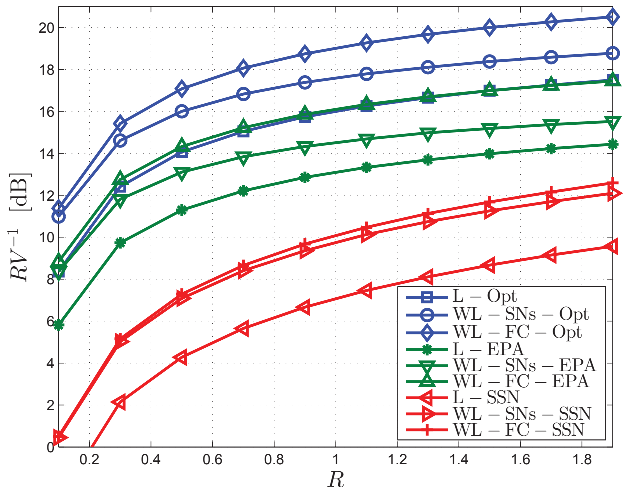

Fig. 2: Estimation accuracy in terms of RV^{-1} [dB] vs. Source signal strength R [Watt].

Appendix

A. Proof to Lemma 1

Firstly, it is observable that (48) does not violate the power constraints as (28) and (29) do not depend on the fusion weights. Secondly, the unbiased condition (31) sill holds since

$$\sum_{k\in\mathbb{F}_K} \bar{v}_k h_k \left(\hat{u}_k g_k + \check{u}_k g_k^*\right) = \sum_{k\in\mathbb{F}_K} \frac{|v_k h_k \left(\hat{u}_k g_k + \check{u}_k g_k^*\right)|}{\sum_{k\in\mathbb{F}_K} |v_k h_k \left(\hat{u}_k g_k + \check{u}_k g_k^*\right)|} = 1, \tag{46}$$

which indicates that $\bar{v}_1, \cdots, \bar{v}_K$ is feasible. And finally, following the triangular inequality we have

$$1 = \left|\sum_{k\in\mathbb{F}_K} v_k h_k \left(\hat{u}_k g_k + \check{u}_k g_k^*\right)\right| \leq \sum_{k\in\mathbb{F}_K} |v_k h_k \left(\hat{u}_k g_k + \check{u}_k g_k^*\right)|, \tag{47}$$

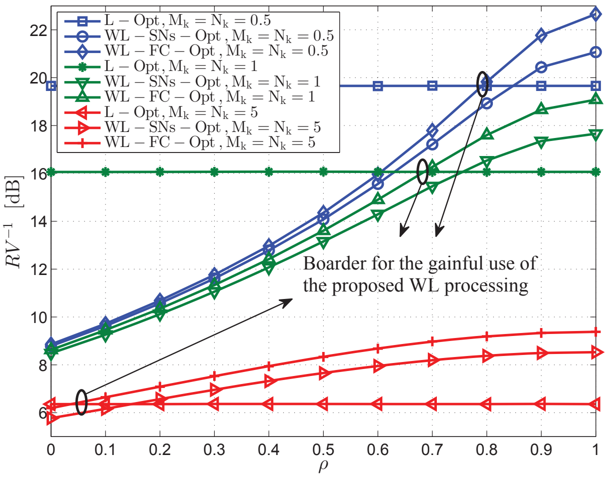

Fig. 3: Estimation accuracy in terms of RV^{-1} [dB] vs. Source signal non-circularity (ρ).

which together with (48) results in $|\bar{v}_k| \leq |v_k|$. The latter inequality reveals that the set of updated fusion weights in (35) does not increase the objective function, which concludes the proof.

B. Proof to Lemma 2

Let $\hat{u}_k, \check{u}_k\ k \in \mathbb{F}_K$ represent a set of optimal amplification coefficients. Then the following variable update:

$$\hat{w}_k \leftarrow \frac{g_k^* |\hat{u}_k g_k + \check{u}_k g_k^*|}{2|g_k|^2},\ \check{w}_k \leftarrow \frac{g_k |\hat{u}_k g_k + \check{u}_k g_k^*|}{2|g_k|^2}, \tag{48}$$

where $\hat{w}_k, \check{w}_k, \forall k \in \mathbb{F}_K$, representing a new set of amplification coefficients, and $\hat{w}_k = \check{w}_k^*$. By examining the different parts of the objective function in (42) we have

$$|\hat{w}_k g_k + \check{w}_k g_k^*| = |\hat{u}_k g_k + \check{u}_k g_k^*|, \tag{49}$$

and

$$\begin{aligned} |\hat{w}_k|^2 + |\check{w}_k|^2 &= \frac{|\hat{u}_k g_k + \check{u}_k g_k^*|^2}{2|g_k|^2} \\ &\leq \frac{|\hat{u}_k|^2 + |\check{u}_k|^2}{2} + |\hat{u}_k \check{u}_k| \\ &\leq |\hat{u}_k|^2 + |\check{u}_k|^2 - \frac{1}{2}\left(|\hat{u}_k| - |\check{u}_k|\right)^2 \\ &\leq |\hat{u}_k|^2 + |\check{u}_k|^2, \end{aligned} \tag{50}$$

which conclude the non-decreasing effect of the defined update on the objective. Furthermore, (49) and (50) show the non-increasing effect of the defined update on the transmit power from each SN, see (28), (29). This shows that the updated set of amplification coefficients, $\hat{w}_k, \check{w}_k, \forall k \in \mathbb{F}_K$, is indeed an optimal solution to (42), as they are both feasible and do not degrade the objective compared to the optimal amplification coefficient set $\hat{u}_k, \check{u}_k, \forall k \in \mathbb{F}_K$. The later argument concludes the proof.

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