Optimal Linear MMSE Design for Passive Distributed Radar Sensor Network Systems

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***Abstract*—In this paper we present optimal power allocation, together with optimal linear signal fusion, considering a passive distributed radar sensor network system. The goal of a passive distributed radar is to obtain a reliable estimation from a source signal, by collecting and combining the individual observations from the network of sensor nodes (SN)s in a fusion center (FC). In this respect, a linear minimum-mean-square-error (MMSE) optimization strategy is considered, where optimal linear operation at the SNs as well as the FC are obtained analytically. The obtained solutions are then analytically and numerically compared to the previously studied unbiased linear MMSE (ULMMSE) approach. It is observed that both schemes share the same strategy for the optimal power allocation among the SNs, but differ in the corresponding linear fusion. As expected, the proposed approach reaches a lower estimation MSE compared to the ULMMSE one.**

***Index Terms*—Wireless Sensor Network, Distributed Radar, MMSE Estimation.**

I. Introduction

Nowadays, many applications benefit from the idea of distributed sensor networks for the purposes of observation and communication. The range of these applications covers health care, traffic monitoring, radio astronomy, particle physics, and military applications [1], [2]. In particular, the goal of a distributed passive radar is to provide a reliable estimation from a source signal, by collecting and combining the individual passive observations from a network of SNs in a centralized node, i.e., FC. As an interesting example, we can mention the '*IceCube Neutrino Observatory*' at the south pole, where a network with more than 5000 SNs is deployed to observe certain characteristics of sub-atomic particles [3]. In this respect, problems regarding the optimal power allocation among the SNs and the energy-aware system design are of interest, generally due to the weak and distributed nature of the SNs, see [1], [4]–[6], and the references therein. In particular, an optimal design of the linear system operation is presented in [1], [7]–[10], employing a ULMMSE optimization framework. The aforementioned works are then extended following a similar optimization strategy, with the consideration of network lifetime and energy efficiency [11]–[14], distributed beamforming among the SNs [15], and occasional node failure and network data inaccuracy [16]. For the sources with strictly non-circular distribution, the gains of a widely linear processing at the SNs, or at the FC is studied in [17].

In this work we extend the proposed designs in [1], [7], [8], which are presenting a ULMMSE approach, to a general linear MMSE (LMMSE) optimization framework. While the imposition of an unbiasedness constraint into the optimization framework is usually favorable for classification applications, an LMMSE approach is widely used for estimation of e.g., noise powers, signal spectrum, fluctuation of temperature, radiation intensity, see [18]. The wide range of applications reinforces the investigation and comparison of both approaches, i.e., ULMMSE and LMMSE, within the framework of radar sensor networks. In particular, we present an optimal linear signal fusion, as well as the optimal power allocation among the SNs. The enhanced estimation accuracy, corresponding to the reduced MSE is analytically observed. At the end, the proposed design is compared to the available ULMMSE solution for different network conditions via numerical simulations.

Mathematical Notations:

Throughout this paper, we denote the sets of complex, real, real and non-negative (non-positive) numbers by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$ ($\mathbb{R}_-$), respectively. The absolute value, square root, complex conjugation, mathematical expectation, and partial derivative are respectively denoted as $|\cdot|$, $\sqrt{\cdot}$, $(\cdot)^*$, $\mathcal{E}\{\cdot\}$, $\partial(\cdot)$. The notation $\bar{c}$ stands for the value of an optimization variable c where the optimum is attained. The set $\mathbb{F}_K$ is defined as $\{1,\ldots,K\}$.

II. System Model

We investigate a network of K amplify-and-forward (AF) passive SNs, cooperating to achieve a single global observation via the FC, see Fig 1. Both communication and sensing channels (frequency-flat fading) are assumed to be wireless and static during the observation process. The final goal of each observation is to estimate a source signal $r \in \mathbb{C}$, at the FC. Each observation can be segmented into three parts: sensing, communication, and information fusion. The detailed function of each SN is discussed in [1, Section II].

A. Operation of SNs

If a source signal $r \in \mathbb{C}$ is present, each SN receives and amplifies the incoming signal using an amplification coefficient $u_k \in \mathbb{C}$. The communication with FC is performed by using orthogonal waveforms for each SN so that the data from different SNs can be separated and processed at the FC[1].

[1]Note that the assumed SN operation fits for the network of SNs with weak processing/cooperation capability, where coordinated transmission schemes, e.g., distributed beamforming schemes, are not feasible, see [19].

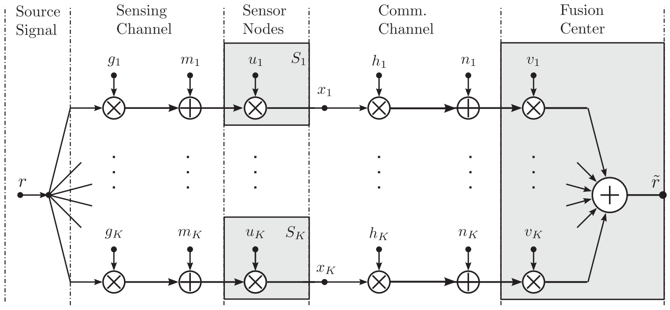

Fig. 1: A passive distributed radar sensor network with K SNs.

The process of each SN can be hence described as

$$x_k := (g_k r + m_k)\, u_k, \;\; k \in \mathbb{F}_K, \tag{1}$$

and

$$X_k := \mathcal{E}\{|x_k|^2\} = \left(R|g_k|^2 + M_k\right)|u_k|^2, \tag{2}$$

where $R := \mathcal{E}\{|r|^2\}$ represents the power of the source signal. The sensing channel coefficient, the transmit signal from the SN with index k and its power are respectively denoted by $g_k \in \mathbb{C}$, $x_k \in \mathbb{C}$ and X_k. The additive white Gaussian noise (AWGN) on the sensing process and its variance are respectively denoted as $m_k \in \mathbb{C}$ and $M_k \in \mathbb{R}_+$. Furthermore, it is assumed that the power consumption of each SN may not exceed a certain limit, namely P_k, where the total average power consumption of the network is limited by P_{tot}:

$$X_k \leq P_k, \; k \in \mathbb{F}_K, \;\; \sum_{k \in \mathbb{F}_K} X_k \leq P_{\text{tot}}. \tag{3}$$

B. Fusion Center

The transmitted signal from each SN passes through the communication channel, with coefficient $h_k \in \mathbb{C}$, and arrives at the FC combined with an AWGN component $n_k \in \mathbb{C}$, with variance $N_k \in \mathbb{R}_+$. A linear combination rule with weights $v_k \in \mathbb{C}$ is then applied at the FC to achieve an estimate, $\tilde{r}$, from the observed source signal. This is described as

$$\tilde{r} := r \sum_{k \in \mathbb{F}_K} v_k h_k u_k g_k + \sum_{k \in \mathbb{F}_K} v_k \left(h_k u_k m_k + n_k\right). \tag{4}$$

where $\tilde{r}$ represents the estimated source signal at the fusion center.

C. Remarks

In the present work, we assume the availability of perfect channel information for both sensing and communication channels. In general, it is rather difficult to estimate the sensing channel in an accurate way unless the channel has a highly stationary nature, see, e.g., [3]. In this respect the sensitivity of the resulting estimation accuracy, via the utilization of the proposed design, to the channel knowledge inaccuracy is numerically studied in Section IV. Moreover, for the scenarios where the sensing channel is not perfectly known, the results of this paper can be treated as theoretical limits.

III. Optimal LMMSE Source Signal Estimation: Linear Signal Fusion and Power Allocation

In this part we obtain the optimal set of the fusion weights v_k, as well as the amplification coefficients u_k, which result in the minimum estimation mean-square-error (MSE) of the source signal at the FC. Following (4), the estimation MSE is formulated as

$$\mathcal{E}\{|\tilde{r} - r|^2\} = \\ R\Big|\underbrace{-1 + \sum_{k \in \mathbb{F}_K} v_k h_k u_k g_k}_{=:W}\Big|^2 + \underbrace{\sum_{k \in \mathbb{F}_K} |v_k|^2 \left(N_k + |h_k u_k|^2 M_k\right)}_{=:V}, \tag{5}$$

as the noise and source signals are all zero-mean and mutually independent. Note that the term V represents the part of the estimation MSE resulting from the zero-mean noise components, similar to that of [1, Eq. (10)], while the additional term $R|W|^2$ represents the part resulting from the estimation bias. The corresponding LMMSE optimization is formulated as

$$\underset{u_k, v_k, k \in \mathbb{F}_K}{\text{minimize}} \;\; V + R|W|^2, \quad \text{s.t. } (3), \tag{6}$$

where (3) imposes the individual, and collective power constraints on the SNs. It is observable that (6) is not a jointly convex optimization problem. Nevertheless, it is separately convex over the amplification coefficients, i.e., u_k, and the fusion weights, i.e., v_k. In the following we obtain the optimal fusion weights for a fixed set of u_k.

A. Optimal LMMSE Fusion

It is observed from (3) that the power constraints are invariant to the choice of v_k. As a result, the corresponding optimization turns into an un-constraint minimization of a convex function

$$\underset{v_k, k \in \mathbb{F}_K}{\text{minimize}} \;\; V + R|W|^2. \tag{7}$$

Due to the convex, non-negative, and differentiable nature of the objective, the optimal fusion coefficients are necessarily (and sufficiently) located at the stationary points. This is formulated, following the guidelines of Wirtinger calculus on complex-valued differentiation [20], [21], as

$$\frac{\partial}{\partial v_k}\left(V + R|W|^2\right) = 0 \;\; \Leftrightarrow \\ RW^* h_k u_k g_k + v_k^*(N_k + |h_k u_k|^2 M_k) = 0, \tag{8}$$

which results in

$$\bar{v}_k = -WR \frac{h_k^* g_k^* u_k^*}{N_k + |h_k u_k|^2 M_k}. \tag{9}$$

Moreover, by recalling (8), and the fact that

$$\sum_{k \in \mathbb{F}_K} \left(\frac{h_k^* g_k^* u_k^*}{N_k + |h_k u_k|^2 M_k}\right) \frac{\partial}{\partial v_k}\left(V + R|W|^2\right) = 0, \tag{10}$$

we have

$$\sum_{k \in \mathbb{F}_K} RW^* \frac{|h_k g_k u_k|^2}{N_k + |h_k u_k|^2 M_k} + \sum_{k \in \mathbb{F}_K} h_k^* g_k^* u_k^* v_k^* = 0, \tag{11}$$

and consequently from (5) it follows that

$$\bar{W} = -\left(1 + R\sum_{k\in\mathbb{F}_K} \frac{|h_k g_k u_k|^2}{N_k + |h_k u_k|^2 M_k}\right)^{-1}, \tag{12}$$

which also concludes that $\bar{W} \in \mathbb{R}_-$. Moreover, from (12), we have

$$\bar{v}_k = \left(\frac{1}{R} + \sum_{k\in\mathbb{F}_K} \frac{|h_k g_k u_k|^2}{N_k + |h_k u_k|^2 M_k}\right)^{-1} \frac{h_k^* g_k^* u_k^*}{N_k + |h_k u_k|^2 M_k}, \tag{13}$$

which presents a closed form expression for the optimal fusion coefficients, given a fixed (and feasible) set of u_k. Similar to the previous steps, and considering

$$\sum_{k\in\mathbb{F}_K} v_k \frac{\partial}{\partial v_k}\left(V + RW^2\right) = 0, \tag{14}$$

we have

$$\sum_{k\in\mathbb{F}_K} RW h_k g_k u_k v_k + \sum_{k\in\mathbb{F}_K} |v_k|^2 \Big(N_k + |h_k u_k|^2 M_k\Big) = 0, \tag{15}$$

and consequently it follows that

$$\bar{V} = -R\bar{W}(\bar{W} + 1). \tag{16}$$

Moreover from (5) we obtain $\bar{\text{MSE}} = -R\bar{W}$, which results in

$$\bar{\text{MSE}} = \left(\frac{1}{R} + \sum_{k\in\mathbb{F}_K} \frac{|h_k g_k u_k|^2}{N_k + |h_k u_k|^2 M_k}\right)^{-1}, \tag{17}$$

where $\bar{V}, \bar{\text{MSE}}$ respectively represent the obtained values of V and the estimation MSE, via the application of the optimal fusion coefficients (13).

B. Optimal Linear SN Operation: Power Allocation Problem

For a fixed set of u_k, the results in (13) and (17) respectively represent the optimal linear fusion, and the corresponding MSE value. The remaining task is hence to find a set of u_k, that minimizes the resulting $\bar{\text{MSE}}$, while satisfying the power constraints defined in (3). This is formulated as

$$\underset{u_k,\, k\in\mathbb{F}_K}{\text{minimize}}\ \bar{\text{MSE}}, \quad \text{s.t. } (3). \tag{18}$$

Note that the phase of the amplification coefficients u_k do not have an effect on the resulting $\bar{\text{MSE}}$, as it is apparent from (17)[2]. Consequently we assume $u_k \in \mathbb{R}_+$, withought loss of generality. By formulating $u_k = \sqrt{\frac{X_k}{R|g_k|^2 + M_k}}$, as a result of (1) and (2), the above problem can be equivalently formulated as

$$\underset{X_k\in\mathbb{R}_+, k\in\mathbb{F}_K}{\text{maximize}} \quad \sum_{k\in\mathbb{F}_K} \frac{\alpha_k^2 X_k}{X_k + \beta_k^2} \tag{19a}$$

$$\text{s.t.} \quad \sum_{k\in\mathbb{F}_K} X_k \le P_{\text{tot}}, \quad X_k \le P_k,\ k\in\mathbb{F}_K, \tag{19b}$$

[2]This is grounded in two reasons. Firstly, the phase of u_k does not impact the power consumption at the corresponding SN, and hence does not impact the power constraints (3). Secondly, any phase of u_k is eventually canceled out by the phase of the optimally-chosen fusion coefficients v_k, and will not impact the resulting estimate $\tilde{r}$, see (4) in connection with (13).

where $\alpha_k := \sqrt{\frac{|g_k|^2}{M_k}}$, and $\beta_k := \sqrt{\frac{N_k(R|g_k|^2 + M_k)}{M_k|h_k|^2}}$ are formulated similar to [1, Eq. (22), (23)]. Interestingly, the problem of optimizing the amplification coefficients (19) coincides with the studied similar problem for ULMMSE scheme, which results in a water-filling solution structure. For an analytical optimum solution to (19), and further elaborations see [1, Eq. (35)] or [16, Eq. (20)].

C. Observations: Best LMMSE vs. ULMMSE Estimators

In this part we observe the impact of the proposed LMMSE design, compared to the previously studied ULMMSE case.

1) Optimum amplification and power allocation: It is observed from (19) that the proposed LMMSE approach results in a similar optimum power allocation solution as to the studied ULMMSE case, see (19) in comparison with [1, Eq. (21), (46)]. This shows that the quality of the SNs, in terms of the allocation of the available resources, do not depend on the estimation strategy but on their channel and noise conditions. Furthermore, it enables the FC to customize the appropriate fusion strategy to a specific application, with no need to adapt the optimal operation of the SNs[3].

2) Estimation Accuracy: As expected, the resulting estimation MSE via the utilization of the LMMSE estimator is always smaller in comparison with the unbiased case, see (17) in comparison with [1, Eq. (21)]. In particular, the resulting estimation accuracy in terms of R/MSE can be directly compared as[4]:

$$\begin{aligned} R/\bar{\text{MSE}} &= 1 + R\sum_{k\in\mathbb{F}_K} \frac{|h_k g_k u_k|^2}{N_k + |h_k u_k|^2 M_k} \\ &= 1 + R/\text{MSE}_{\text{unbiased}}, \end{aligned} \tag{20}$$

verifying the known ULMMSE-LMMSE performance gap [22], where $\text{MSE}_{\text{unbiased}}$ represents the resulting estimation MSE via the application of the optimum ULMMSE fusion, see (17) and (12) in comparison with [1, Eq. (21)]. Please note that there is no closed form expression for the final MSE after the utilization of the jointly optimal u_k, v_k, due to the water-filling optimal solution structure for u_k. Nevertheless, as both schemes result in the same optimal choice of the amplification coefficients, this gap holds for the jointly optimal choice of u_k, v_k, as well.

IV. Simulation Results

In this part we evaluate the performance of the proposed estimator, in comparison to the studied ULMMSE estimator in [1], [7], via numerical simulations. Fig. 2 represents the simulated setup, where a network of 12×12 nodes is simulated. Small circles, triangular, and rectangular shapes respectively represent the SNs, the source and FC locations. The distance of

[3]An interesting use case can be a hybrid usage of LMMSE and ULMMSE strategies at the FC. In this case the ULMMSE can be used for detecting the presence of a source, and once the source is known to be present, the LMMSE fusion can be used to obtain a lower MSE. As mentioned, the optimal operation of the SNs remains the same for both cases.

[4]The ratio R/MSE compares the variance of the (to be estimated) source signal, i.e., R, to the estimation MSE.

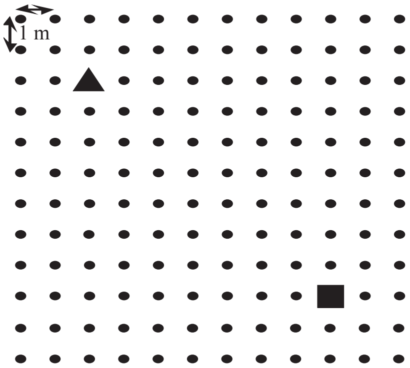

Fig. 2: The simulated setup: small circles, triangular, and rectangular shapes respectively represent the SNs, the source and FC locations.

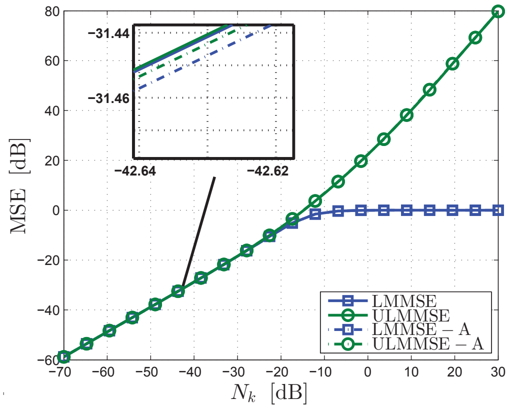

Fig. 3: Estimation MSE vs. $M_k = N_k$. Estimation accuracy degrades as noise variance increases.

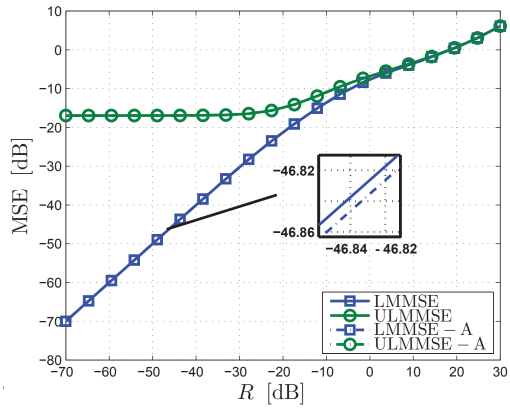

Fig. 4: Estimation MSE vs. R. Estimation MSE increases as source signal variance increases.

two adjacent nodes is 1 meter. All sensing and communication channel coefficients are generated with zero-mean complex Gaussian distribution. The variance of the channel coefficients between each two nodes is determined as $d^{-\zeta}$, where d represents the distance, and ζ is the path loss exponent. The default values for the network parameters is presented in Table I. For each set of the channel realizations, i.e, h_k, g_k, the resulting u_k, v_k are calculated via the best ULMMSE [1], and

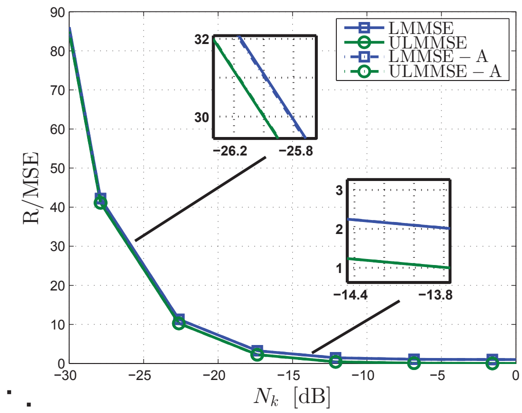

Fig. 5: R/MSE vs. $M_k = N_k$. Estimation accuracy degrades as noise variance increases.

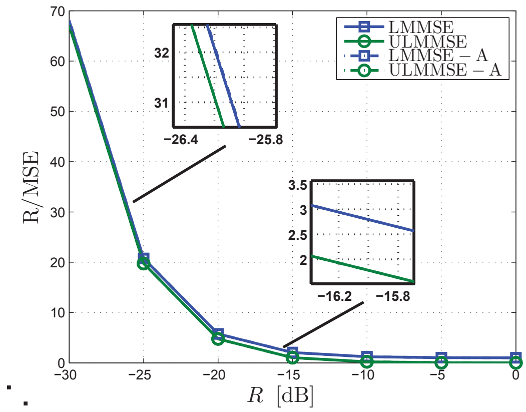

Fig. 6: R/MSE vs. R. Estimation accuracy degrades as source signal strength increases.

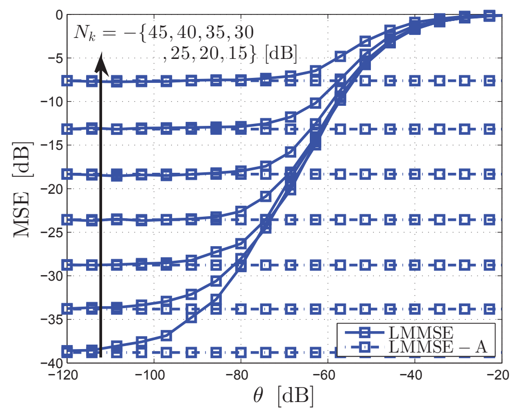

Fig. 7: MSE vs. θ. Estimation accuracy degrades as as CSI accuracy degrades.

the proposed LMMSE estimation strategies. Afterwards, 1000 realizations of r, n_k, m_k are generated in order to evaluate the resulting estimation MSE. The resulting MSE is then averaged over 1000 channel realizations. The legends ULMMSE and LMMSE represent the aforementioned numerically evaluated

TABLE I: Default Values

R	P_k	P_{tot}	ζ	M_k	N_k
1	1	10	3	0.1	0.1

performance, while the legends LMMSE-A and ULMMSE-A represent the analytically obtained MSE respectively from (17) and [1, Eq. (21)].

In Fig. 3 and Fig. 4, the resulting estimation MSE is illustrated for different noise levels $M_k = N_k$, and target signal power, i.e., R. It is observed that the analytically expected performance is accurately followed by the numerical evaluations. Moreover, while the increase of noise intensity increases the resulting MSE for both schemes, the LMMSE shows a higher robustness as the noise level increases.

In Fig. 5 and Fig. 6 the resulting estimation accuracy, in terms of R/MSE is illustrated for the similar schemes as in Fig. 3 and in Fig. 4. The analytically obtained constant gap in (20) is observed between the best LMMSE and the ULMMSE estimators.

In Fig. 7 the sensitivity of the proposed estimators to the channel estimation error is studied. The channel uncertainty is modeled as $\tilde{g}_k = g_k + \delta_k$, and $\tilde{h}_k = h_k + \epsilon_k$, where $\tilde{g}_k, \tilde{h}_k$ are the estimated versions of the sensing and communication channels, and δ_k, ϵ_k are estimation errors, modeled as zero-mean complex Gaussian random process, with variance θ[5]. It is observed that the resulting estimation MSE converges to the analytically expected values as the error variance is small, while converging to 0 [dB] as θ increases. Moreover, the resulting performance enjoys a higher level of robustness for a system with a relatively higher noise level, which is commonly the case for the distributed SN systems. Conversely, as the noise variance decreases, a higher sensitivity is observed and hence a better estimation accuracy is required.

V. Conclusion

In this paper the optimal linear signal fusion, as well as the power allocation on the SNs is obtained for a network of distributed passive radar system, following an LMMSE approach. It is observed that the optimal power allocation on the SNs coincides with that of the studied ULMMSE approach. Nevertheless, the obtained optimal signal fusion in the LMMSE case reaches a lower level of the estimation MSE, as expected. Moreover, it is observed that the channel estimation inaccuracy degrades the performance of the proposed design. Nevertheless, this error can be better tolerated for a system with high noise level, which is usually the case for the distributed sensor network systems.

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[5] The estimated channel coefficients are used for designing the estimators. Nevertheless, the actual, i.e., error-free, channel coefficients are used for evaluating the resulting MSE.