Feasibility and Power Control for Linear Multiuser Receivers in CDMA Networks

Daniel Catrein, *Member, IEEE,* and Rudolf Mathar, *Member, IEEE*

***Abstract*—This paper is concerned with linear multiuser receivers for code division multiple access (CDMA). Mainly three are of practical relevance: the matched filter (MF), the successive interference cancellation (SIC) and the minimum mean squared error (MMSE) receiver. For the first time, an explicit representation of the signal-to-interference-plus-noise (SINR) ratio for the linear SIC receiver under general random spreading is given. For the MMSE receiver the SINR is obtained by using the asymptotic large system regime as an accurate approximation. Once the SINR for each receiver type is known, the question arises what quality-of-service (QoS) requirements can be supported by an adequate power assignment. A vector of QoS requirements (γ_i) is called feasible, if there exists some power assignment such that the SINR of each user i does not fall short of threshold γ_i. It is shown that for each receiver type there exists a componentwise minimal power assignment such that the SINR of each user equals this threshold. This minimal vector may be determined for the MF and SIC receiver by solving a system of linear equations with coefficients explicitly known. For the MMSE receiver in the large system regime an iterative algorithm is derived, which converges to the optimum power allocation.**

***Index Terms*—Matched filter, successive interference cancellation, minimum mean squared error, random spreading sequences, large system regime, standard interference function, large system regime.**

I. Introduction

FEASIBILITY and power control for code division multiple access (CDMA) radio networks are closely related. On one hand, the question arises whether for a given set of signal-to-interference-plus-noise thresholds there exists a power assignment such that each user's SINR meets this threshold. On the other hand, once having clarified that a solution exists, for practical purposes there is need for methods to compute the power minimal solution.

Both questions are well investigated in the literature, however mainly for the matched filter receiver, see, e.g., [1], [2]. A comprehensive study, including significantly improved algorithms and an axiomatic embedding of the field of resource allocation is given in the up to date books [3], [4]. The recent conference contribution [5] deals with SIC/MMSE detection and also uses the approximate large system regime introduced in [6]. New formulas are given for the system SINR and power control algorithms are derived.

Although feasibility and power control are currently well investigated, there is a number of open problems mainly for the SIC/MF and the MMSE receiver. It is the purpose of the present paper to introduce a unifying theory of feasibility and power control problems in CDMA for different receiver types: the matched filter (MF), the successive interference cancellation (SIC), and the minimum mean squared error (MMSE) receiver.

The novel contributions in this paper are as follows. We first derive a unified vector representation of the SINR for linear receivers in CDMA. Based on this general form, in Section III an explicit representation of the SINR for SIC detectors and binary random spreading is given. This is derived from an extended result for general spreading sequences that is deferred to a separate section. For analyzing the SINR under MMSE detection we employ the large system regime introduced in [6]. By simulation it is thoroughly analyzed that the deviation between the approximate and exact SINR is actually very small.

In section IV, feasibility of a given vector of QoS thresholds is characterized. It is further shown that for all receiver types there exists a power allocation of uniformly minimal energy which satisfies the QoS requirements with equality, provided there exists any. For the MMSE detector this result is achieved by utilizing the fixed points of the SNR equation and a standard interference function according as defined in [7]. This approach also leads to a convergent algorithm to determine the minimal power assignment for MMSE detectors.

II. System Model

We consider a symbol and chip synchronous CDMA system of N users. The symbol of user i is modeled by a complex valued, zero-mean random variable X_i with expected transmit power $\mathrm{E}(|X_i|^2) = p_i > 0,\ i = 1,\dots,n$. The signature sequences of spreading gain $L \in \mathbb{N}$ are real random vectors

$$\boldsymbol{s}_i = (s_{1i},\dots,s_{Li})^{\mathsf{T}}$$

with all components $s_{ji},\ j = 1,\dots,L,\ i = 1,\dots,N$, stochastically independent. In the sequel we assume that the random variables $\boldsymbol{s}_i$ have expected squared norm equal to one, that is,

$$\mathrm{E}\left(\boldsymbol{s}_i^{\mathsf{T}}\boldsymbol{s}_i\right) = 1,\ i = 1,\dots,N. \tag{1}$$

Condition (1) is obviously satisfied by definition if $\boldsymbol{s}_i$ has unity norm with probability one, that is,

$$P(\boldsymbol{s}_i^{\mathsf{T}}\boldsymbol{s}_i = 1) = 1,\ i = 1,\dots,N. \tag{2}$$

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D. Catrein was with the Institute of Theoretical Information Technology, RWTH Aachen University when working on the results presented here. He is now with Ericsson Research, Ericsson GmbH, Herzogenrath, Germany (e-mail: daniel.catrein@ericsson.com).

R. Mathar is with the Institute for Theoretical Information Technology, RWTH Aachen University, Aachen, Germany (e-mail: mathar@ti.rwth-aachen.de).

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Binary random spreading with

$$P(s_{ji} = L^{-1/2}) = P(s_{ji} = -L^{-1/2}) = \frac{1}{2} \tag{3}$$

for all $j = 1, \dots, L$ and all users $i = 1, \dots, N$ is a special case hereof. It should be noted that random spreading sequences are not necessarily orthogonal.

In the above framework, deterministic spreading sequences may be modeled by a singleton distribution satisfying $P(\boldsymbol{s}_i = \dot{\boldsymbol{s}}_i) = 1$ for a set of given sequences $\{\dot{\boldsymbol{s}}_1, \dots, \dot{\boldsymbol{s}}_N\}$ satisfying $\|\dot{\boldsymbol{s}}_i\| = 1$. The sequences $\dot{\boldsymbol{s}}_i$ should be chosen pairwise orthogonal if $N \le L$.

The channel attenuation for user i is described by stochastically independent complex random variables H_i with $\mathrm{E}(|H_i|^2) = a_i$, the channel gain coefficient.

Neglecting the time dependence and the detailed structure of the chip waveforms the received signal may be described by a vector $\boldsymbol{Y}$ as

$$\boldsymbol{Y} = \sum_{i=1}^{N} \boldsymbol{s}_i H_i X_i + \boldsymbol{W},$$

where $\boldsymbol{W} = (W_1, \dots, W_N)^{\mathsf{T}}$ is a complex, zero mean Gaussian random vector with uncorrelated components W_i satisfying $\mathrm{E}(W_i) = \sigma^2$, independent of all other random variables involved.

Linear receivers are now described by random vectors $\boldsymbol{c}_1, \dots, \boldsymbol{c}_N \in \mathbb{R}^L$. For each user, the estimator $\hat{X}_i$ of the transmitted symbol X_i is given by

$$\hat{X}_i = \boldsymbol{c}_i^{\mathsf{T}} \boldsymbol{Y}. \tag{4}$$

Different types of linear multiuser receivers are obtained by specific choices of $\boldsymbol{c}_1, \dots, \boldsymbol{c}_N$ as functions of $\boldsymbol{s}_1, \dots, \boldsymbol{s}_N$.

In general, the signal-to-interference-plus-noise ratio for user i is defined as

$$\mathrm{SINR}_i = \frac{\mathrm{E}\left(|\boldsymbol{c}_i^{\mathsf{T}} \boldsymbol{s}_i H_i X_i|^2\right)}{\mathrm{E}\left(\left|\sum_{j \ne i} \boldsymbol{c}_i^{\mathsf{T}} \boldsymbol{s}_j H_j X_j + \boldsymbol{c}_i^{\mathsf{T}} \boldsymbol{W}\right|^2\right)}. \tag{5}$$

Because of the independence assumptions, (5) can be further simplified to

$$\mathrm{SINR}_i = \frac{\mathrm{E}\left((\boldsymbol{c}_i^{\mathsf{T}} \boldsymbol{s}_i)^2\right) a_i p_i}{\sum_{j \ne i} \mathrm{E}\left((\boldsymbol{c}_i^{\mathsf{T}} \boldsymbol{s}_j)^2\right) a_j p_j + \mathrm{E}\left(\boldsymbol{c}_i^{\mathsf{T}} \boldsymbol{c}_i\right) \sigma^2}, \tag{6}$$

where $a_i = \mathrm{E}(|H_i|^2)$ are the channel gain coefficients.

In the case of deterministic spreading and deterministic linear receivers, the expectation signs in (6) may be simply omitted, assuming that the singleton distribution concentrates its mass on given $\boldsymbol{c}_i$ and $\boldsymbol{s}_i$, respectively.

Receiver vector $\boldsymbol{c}_i$ is called optimal for user i if it maximizes the SINR_i of user i.

The uplink of many commercial CDMA systems can be modeled using pseudo random sequences. For example, in the uplink of a UMTS system, the low rate data signal is first spread by multiplication with a deterministic spreading sequence to a common chip rate. Afterward, the spread data signal is multiplied with pseudo random scrambling sequences, which are unique to each user. For implementation purposes, spreading and scrambling are commonly considered to be separated. To ease the theoretical model, spreading and scrambling can be modeled as a single operation together. That is, the data signal is spread with a pseudo random spreading sequence constructed by taking deterministic spreading and successive random scrambling as a single black-box operation; its spreading gain corresponding to that of the deterministic spreading. [1] Thus, the SINR of this receiver is of tremendous practical interest.

III. SINR for Specific Linear Receivers

In this section, we further investigate the SINR for three different receiver types and compare their performance.

A. Matched Filter Receiver

Matched filter receivers use the receiver vectors

$$\boldsymbol{c}_i^{\mathrm{MF}} = \boldsymbol{s}_i, \ i = 1, \dots N.$$

The matched filter receiver is the optimal linear receiver for a single user channel with uncorrelated Gaussian noise, see [8].

From assumption (2) it follows that $\mathrm{E}(\boldsymbol{s}_i^{\mathsf{T}} \boldsymbol{s}_i) = 1$, and also $\mathrm{E}\left((\boldsymbol{s}_i^{\mathsf{T}} \boldsymbol{s}_i)^2\right) = 1$. For the matched filter receiver, SINR (6) then simplifies to

$$\mathrm{SINR}_i^{\mathrm{MF}} = \frac{a_i p_i}{\sum_{j \ne i} \mathrm{E}\left((\boldsymbol{s}_i^{\mathsf{T}} \boldsymbol{s}_j)^2\right) a_j p_j + \sigma^2}. \tag{7}$$

Again, for deterministic spreading sequences $\boldsymbol{s}_1, \dots, \boldsymbol{s}_N$ the expectation sign in (7) my be simply omitted.

For binary random spreading (3) it may be easily verified that $\mathrm{E}\left((\boldsymbol{s}_i^{\mathsf{T}} \boldsymbol{s}_j)^2\right) = \frac{1}{L}$. for all $i \ne j \in \{1, \dots, N\}$. Hence, the SINR reduces to

$$\mathrm{SINR}_i^{\mathrm{MF}} = \frac{a_i p_i}{\frac{1}{L} \sum_{j \ne i} a_j p_j + \sigma^2}. \tag{8}$$

B. Successive Interference Cancellation Receiver

The basic idea behind the successive interference cancellation receiver (SIC) is to decode the user signals subsequently and to subtract already decoded signals from the overall received signal. This results in an interference reduction for later decoded users. The SIC concept was first suggested in [9], [10], [11].

Successive interference cancellation is described by the system of equations

$$\hat{X}_i = \boldsymbol{s}_i^{\mathsf{T}} \Big(\boldsymbol{Y} - \sum_{j=1}^{i-1} \boldsymbol{s}_j \hat{X}_j \Big), \ i = 1, \dots, N. \tag{9}$$

Here, $\hat{X}_i$ denotes the estimated symbol of user i, where the empty sum for $i = 1$ is set to zero. Obviously, the receiver is equivalent to the MF receiver for the first user. Subsequent users again employ the MF receiver, however, the estimated interference from previous users is deducted from the overall signal.

[1] Note that in reality the uplink of most practical CDMA systems is neither chip nor symbol synchronous. However, we define the SINR via the expected values. Hence, we expect our results to be, at least qualitatively, transferable to a system without synchronisation.

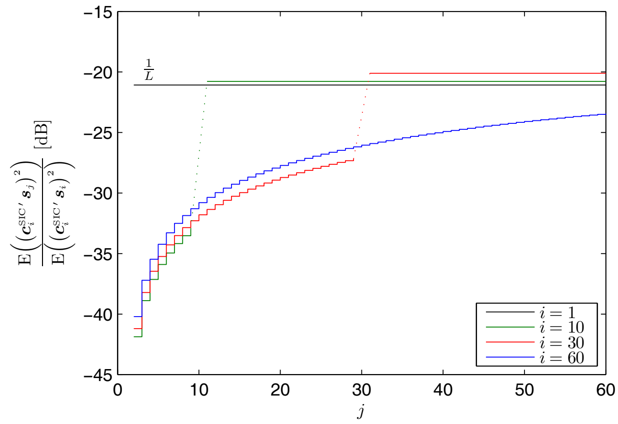


Fig. 1. Interference reduction α_{ij}/α_{ii} for user $j < i$ and constant interference increase $\frac{1}{L}\frac{\beta_i}{\alpha_{ii}}$ for users $j > i$ at spreading gain $L = 128$ in dB.

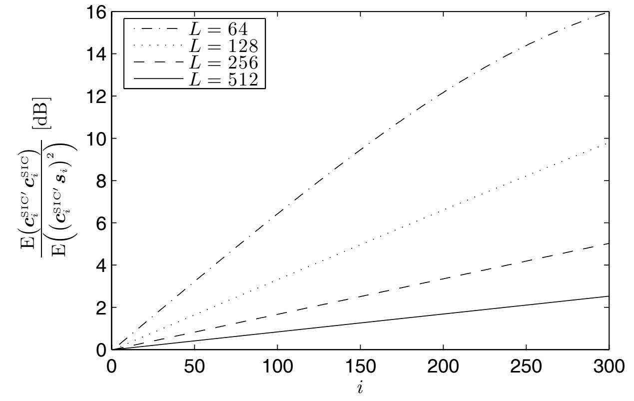


Fig. 2. Noise rise β_i/α_{ii} of user i in dB for spreading length $L \in \{64, 128, 256, 512\}$.

The SIC receiver defined by (9) is shown to be a linear in [12], with an explicit form given as

$$c_i^{\mathrm{SIC}} = \Big(\prod_{j=1}^{i-1} \big(\boldsymbol{I} - \boldsymbol{s}_j \boldsymbol{s}_j^{\mathsf{T}} \big) \Big) \boldsymbol{s}_i, \ i = 1, \dots, N. \qquad (10)$$

Here, the empty product is set to be the identity matrix. Note that linearity may only hold under idealized conditions. Linearity may vanish due to detection and decision errors and due to their propagation. For example, it is shown in [8], that the SIC receiver is not linear if hard-decision variables through the according signum operation are used.

As a major result we will derive an explicit form of the SINR for the SIC receiver. In the literature so far, an approximation is widely adopted which summarizes cross correlation effects by a cancellation imperfectness factor, see [10], [13]. For reasons of readability and easy interpretation we first confine ourselves to binary random spreading. The result for general spreading sequences under the weaker assumption that s_{ji} are independent identically distributed with $\mathrm{E}(s_{ji}) = 0$ and $\mathrm{E}(\boldsymbol{s}_i^{\mathsf{T}} \boldsymbol{s}_i) = 1$ and its full proof will be given in Section V.

Theorem 1: For binary random spreading codes (3) the SINR of user $i = 1, \dots, N$ with a successive interference cancellation receiver is given by

$$\mathrm{SINR}_i^{\mathrm{SIC}}(\boldsymbol{p}) = \frac{a_i\, p_i}{\sum_{j=1}^{i-1} \frac{\alpha_{ij}}{\alpha_{ii}} a_j\, p_j + \frac{1}{L}\frac{\beta_i}{\alpha_{ii}} \sum_{j=i+1}^{N} a_j\, p_j + \frac{\beta_i}{\alpha_{ii}} \sigma^2}. \qquad (11)$$

The constants $\alpha_{ij} = \mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}} \boldsymbol{s}_j\right)^2$ and $\beta_i = \mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}} \boldsymbol{c}_i^{\mathrm{SIC}}\right)$, $i, j = 1, \dots, N$, $i \geq j$, are given with $\omega_L = 1 - \frac{1}{L}$ as

$$\beta_i = \omega_L^{i-1}$$

and

$$\alpha_{ij} = \begin{cases} \omega_L \left(1 - \frac{2}{L} + \frac{2}{L^2}\right)^{i-1} + \frac{1}{L}\omega_L^{i-1}, & \text{if } i = j, \\ \frac{1}{L}\omega_L^{i-j}\left(\omega_L^{j-1} - (1 - \frac{2}{L} + \frac{2}{L^2})^{j-1}\right), & \text{if } i > j. \end{cases}$$

The ratios

$$\frac{\alpha_{ij}}{\alpha_{ii}} = \frac{\mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}} \boldsymbol{s}_j\right)^2}{\mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}} \boldsymbol{s}_i\right)^2}, \ j < i = 2, \dots, N, \qquad (12)$$

and

$$\frac{\beta_i}{\alpha_{ii}} = \frac{\mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}} \boldsymbol{c}_i^{\mathrm{SIC}}\right)}{\mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}} \boldsymbol{s}_i\right)^2} \qquad (13)$$

play a prominent role in (11) when compared to the matched filter receiver (8). For user i, the interference reduction by users $j < i$ already decoded so far is described by (12), $\frac{1}{L}\frac{\beta_i}{\alpha_{ii}}$ represents the interference increase for users to be subsequently decoded, and (13) gives the noise rise in the SINR for user i.

Fig. 1 depicts interference reduction (12) for user $j < i$ and the constant interference increase (13) above $\frac{1}{L}$ for users $j > i$ at spreading gain $L = 128$.

Fig. 2 shows the noise rise (13) for user i and different spreading gains $L \in \{64, 128, 256, 512\}$.

Both quantities are highly variable. Substitution by an approximating constant, as applied in, e.g., [10], [13], may hence lead to model inaccuracies.

C. Minimum Mean Square Error Receiver

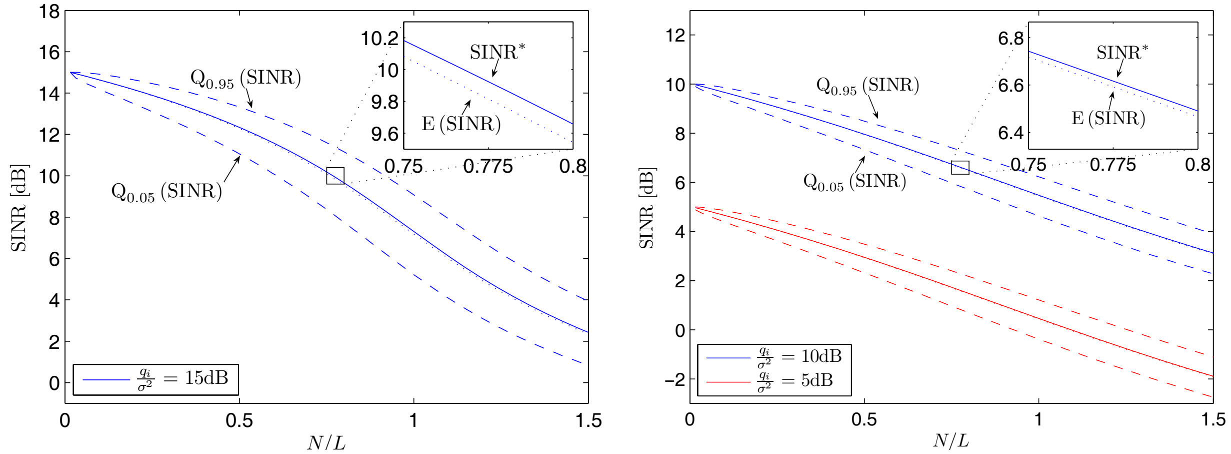
By definition, the minimum mean square error (MMSE) receiver is the optimal linear receiver. The direct analysis of its SINR is rather complicated as it, e.g., requires inversions of random matrices of hitherto unknown distribution. Hence, we employ an asymptotic method developed in [6], the so-called large system analysis. In a CDMA system with processing gain and number of users both increasing with their ratio fixed to $N/L = r$, and with random unit-norm, zero-mean signature sequences, the SINR of each user converges in probability to a constant.

More precisely, we use the following approximate model for the MMSE receiver. As in [6], we assume the SINR of user i is given by the solution $\mathrm{SINR}_i^{\mathrm{MMSE}}(\boldsymbol{p})$ of the fixed point equation

$$\mathrm{SINR}_i = \frac{a_i\, p_i}{\frac{1}{L}\sum_{j \neq i} \frac{a_i p_i a_j\, p_j}{a_i p_i + \mathrm{SINR}_i\, a_j p_j} + \sigma^2}. \qquad (14)$$

For a detailed motivation and derivation we refer to [6].

It is the main purpose of this section to provide a justification of the approximate model (14) by accurate and extensive



(a) Spreading factor $L = 64$, $a_i\,p_i/\sigma^2 = 15$dB for all users.

(b) Spreading factor $L = 128$, $a_i\,p_i/\sigma^2 = 5$dB and $a_i\,p_i/\sigma^2 = 10$dB for the first and the second half of the users, respectively.

Fig. 3. Comparison of the SINR of an MMSE receiver for the exact solution obtained by simulation and the approximation (14).

simulation. Figure 3 demonstrates that this approximation is quite accurate even for small values of L. The approximative $\mathrm{SINR}_i^{\mathrm{MMSE}}$ is compared with the exact values by Monte-Carlo simulations for binary random spreading. Each user i has a constant received power $a_i\,p_i$, expressed in terms of the background noise power σ^2 by the ratio $a_i\,p_i/\sigma^2$. The empirical mean value $\mathrm{E}(\mathrm{SINR}_i)$ and also the empirical quantiles $Q_{0.05}(\mathrm{SINR}_i)$ and $Q_{0.95}(\mathrm{SINR}_i)$, defined by

$$P\big(\,\mathrm{SINR}_i \le Q_p(\mathrm{SINR}_i)\big) = p$$

for $p \in \{0.05, 0.95\}$, are estimated during the simulations.

The results are depicted for a single class of users. All users have a constant received power of 15dB in terms of the background noise. The empirical mean and the estimated quantiles from the simulations are plotted as functions of $N/L = r$ with constant spreading gain $L = 64$. The difference between the empirical mean of the SINR and the approximate value is about 0.1dB. As expected, the approximation gets better for larger L as can be seen from Figure 3(b) for $L = 128$. Half of the users in Figure 3(b) has a constant received power of 10dB and the other half has a received power of 5dB, again in terms of the background noise σ^2. Obviously, different powers for different classes of users do not downgrade the approximation. Moreover, 90% of all observed SINR values lie in an interval of 1.5dB around the estimated exact mean value for $L = 64$. Hence, (14) may be used as an approximation even for a single random SINR realization of an MMSE receiver with an error of the order 1.5dB. Again, longer spreading codes ensue a better approximation and different power requirements have no noticeable effect, see Figure 3(b).

IV. Feasibility and Power Control

The SINR of both the MF and the SIC receiver fall into the same class of functions

$$\mathrm{SINR}_i^{\boldsymbol{\Psi}}(\boldsymbol{p}) = \frac{\psi_{ii}\,p_i}{\sum_{j\neq i}\psi_{ij}\,p_j + \sigma_i^2},\ \ i = 1,\dots,N, \qquad (15)$$

for some nonnegative matrix $\boldsymbol{\Psi} = \big(\psi_{ij}\big)_{1\le i,j\le N}$ with positive diagonal elements $\psi_{ii} > 0$ for all i and $\sigma_i^2 > 0$.

The MF receiver is obtained by choosing $\psi_{ij} = a_j/L$ for $i, j = 1,\dots,N$, $i \neq j$ and $\psi_{ii} = a_i$ otherwise. The SIC receiver SINR is derived by setting $\psi_{ii} = a_i$ and

$$\psi_{ij} = \begin{cases} \frac{\alpha_{ij}a_j}{\alpha_{ii}}, & \text{if } j < i \\ \frac{\beta_i a_j}{L\alpha_{ii}}, & \text{if } j > i \end{cases},$$

with α_{ij}, β_i from Theorem 1. Many other receivers and channel models fall within the scope of equation (15). Even self-interference in the case that the transmitted signal of user i contributes partially to the interference of himself can be included by modifying the requirement vector $\boldsymbol{\gamma}$, see [14].

In view of (15), a given vector $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_N)^\mathsf{T}$ of SINR requirements is called feasible if there is a power vector $\boldsymbol{p} = (p_1,\dots,p_N)^\mathsf{T}$ such that

$$\mathrm{SINR}_i^{\boldsymbol{\Psi}}(\boldsymbol{p}) \ge \gamma_i \text{ for all } i = 1,\dots,N.$$

Denote by

$$\mathcal{P}_{\mathrm{SINR}}^{\boldsymbol{\Psi}}(\boldsymbol{\gamma}) = \big\{\boldsymbol{p} \ge 0 \mid \mathrm{SINR}_i^{\boldsymbol{\Psi}}(\boldsymbol{p}) \ge \gamma_i,\ i = 1,\dots,N\big\}$$

the set of admissible power allocations.

It is shown in [15] that if $\mathcal{P}_{\mathrm{SINR}}^{\boldsymbol{\Psi}}(\boldsymbol{\gamma})$ is nonempty then it has a componentwise minimal element and forms a shifted cone. More precisely, there is a unique power allocation $\boldsymbol{p}^* = \boldsymbol{p}^*(\boldsymbol{\gamma})$ such that

$$\mathrm{SINR}_i^{\boldsymbol{\Psi}}(\boldsymbol{p}^*) = \gamma_i \text{ for all } i = 1,\dots,N, \qquad (16)$$

and $\boldsymbol{p}^* \le \boldsymbol{p}$ for all $\boldsymbol{p} \in \mathcal{P}_{\mathrm{SINR}}^{\boldsymbol{\Psi}}(\boldsymbol{\gamma})$. Moreover, the set $\mathcal{P}_{\mathrm{SINR}}^{\boldsymbol{\Psi}}(\boldsymbol{\gamma}) - \boldsymbol{p}^*(\boldsymbol{\gamma})$ is a closed convex cone.

In matrix form, $\boldsymbol{p}^*$ is given as the positive solution to

$$(\boldsymbol{I} - \mathrm{diag}(\boldsymbol{\gamma})\,\boldsymbol{B})\,\boldsymbol{p} = \mathrm{diag}(\boldsymbol{\gamma})\,\boldsymbol{t}. \qquad (17)$$

with $\boldsymbol{B} = (b_{ij})_{1\le i,j\le N}$, $b_{ij} = \psi_{ij}/\psi_{ii}$ for $i \neq j$, $b_{ii} = 0$ otherwise and $\boldsymbol{t} = (t_1,\dots,t_N)^\mathsf{T}$ with $t_i = \sigma_i^2/\psi_{ii}$. Notation $\mathrm{diag}(\boldsymbol{\gamma})$ means the diagonal matrix with entries γ_i. As is well known, a solution of (17) exists if and only if the spectral radius $\rho\big(\,\mathrm{diag}(\boldsymbol{\gamma})\,\boldsymbol{B}\big) < 1$, see, e.g., [2].

We are mainly interested in $\boldsymbol{p}^*(\boldsymbol{\gamma})$ since it is an admissible power allocation which represents the optimal energy efficient solution.

A. Matched Filter Receiver

For the MF receiver with random spreading the solution $\boldsymbol{p}^*$ to the optimal power control problem with quality constraints has an explicit solution, as is demonstrated in [1], [16].

Proposition 2: The solution to the optimal power control problem for the matched filter receiver with binary random spreading, $\mathrm{SINR}_i^{\mathrm{MF}}(\boldsymbol{p}) = \gamma_i$, $i = 1, \ldots, N$, exists if and only if

$$\sum_{i=1}^{N} \frac{1}{1 + L/\gamma_i} < 1. \tag{18}$$

The solution is given by

$$p_i^* = \frac{L\,\sigma^2}{a_i \left(1 + \frac{L}{\gamma_i}\right)\left(1 - \sum_{j=1}^{N} \frac{1}{1+L/\gamma_j}\right)}$$

for all $i = 1, \ldots, N$.

Obviously, because of unlimited available power the existence of a solution is independent of the channel gains a_i and the background noise σ^2.

When both power and spreading sequences are optimized, a necessary and sufficient condition similar to (18) is given in [17].

B. Successive Interference Cancellation Receiver

For the SIC receiver, α_{ij} and β_i from Theorem 1 are substituted in equation (17) to obtain

$$\left(\boldsymbol{I} - \mathrm{diag}(\boldsymbol{\gamma})\,\boldsymbol{B}^{\mathrm{SIC}}\right)\boldsymbol{p} = \mathrm{diag}(\boldsymbol{\gamma})\,\boldsymbol{t}^{\mathrm{SIC}}$$

with $\boldsymbol{B}^{\mathrm{SIC}} = \left(b_{ij}^{\mathrm{SIC}}\right)_{1\le i,j\le N}$,

$$b_{ij}^{\mathrm{SIC}} = \begin{cases} \beta_i/\alpha_{ii}, & \text{if } i<j \\ 0, & \text{if } i=j \\ \alpha_{ij}/\alpha_{ii} & \text{if } i>j \end{cases}.$$

Further, $\boldsymbol{t}^{\mathrm{SIC}} = (t_1^{\mathrm{SIC}}, \ldots, t_N^{\mathrm{SIC}})^{\mathsf{T}}$ with $t_i^{\mathrm{SIC}} = \frac{\beta_i}{\alpha_{ii}}\sigma^2$.

Unlike the MF receiver, an explicit solution is not available here. A positive solution $\boldsymbol{p}^*$ exists if and only if $\rho\left(\mathrm{diag}(\boldsymbol{\gamma})\,\boldsymbol{B}^{\mathrm{SIC}}\right) < 1$, where $\rho(\boldsymbol{A})$ denotes the spectral radius of some matrix $\boldsymbol{A}$. The existence of a solution depends on the spreading code length, the required SINR values and the number of users in the system.

Comparing SIC and MF receiver reveals some interesting insights into the performance of both. Assume for the following that all users have the same QoS requirement $\gamma_i = \gamma$. In this case $\rho\left(\mathrm{diag}(\boldsymbol{\gamma})\,\boldsymbol{B}^{\mathrm{SIC}}\right) < 1$ is equivalent to $\gamma < 1/\rho\left(\boldsymbol{B}^{\mathrm{SIC}}\right)$ such that $1/\rho\left(\boldsymbol{B}^{\mathrm{SIC}}\right)$ is the maximum feasible simultaneous QoS requirement.

In Fig. 4, the maximum feasible SINR requirement $\gamma = 1/\rho\left(\boldsymbol{B}^{\mathrm{SIC}}\right)$ is shown as a function of the number of users N for various spreading code lengths L for the SIC receiver (solid lines). For the MF receiver, the maximum constant $\gamma_i = \gamma$ is determined from (18) as $\gamma = L/(N-1)$, and plotted as dashed lines for comparison purposes. As expected, the maximum SINR of the SIC receiver is larger than that of the MF receiver if the number of users is less than the spreading gain. Both curves seem to intersect at the point $N = L$, a proof of this conjecture still missing. The crossing point happens to be around zero dB. For practical systems an SINR of at least 3 dB is required such that the SIC receiver shows a better performance regardless of the ordering of successive decoding of users.

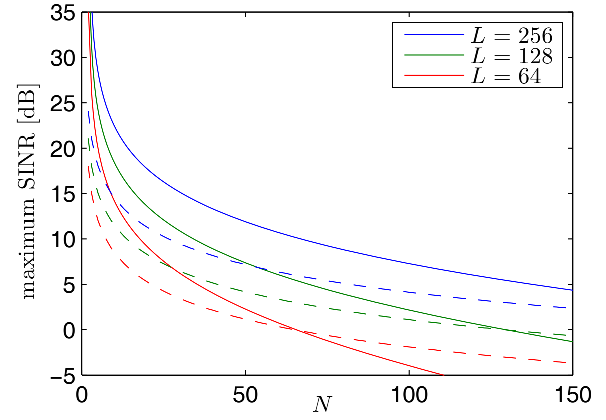


Fig. 4. Maximum SINR for N users in a CDMA system with a SIC receiver (solid line) and a MF receiver (dashed line) in dB for spreading length L.

C. Iterative Power Control

By use of standard interference functions, introduced in [7], we devise algorithms for finding the optimal power assignment. We briefly recall some basic facts from [7].

A function $\boldsymbol{f} = (f_1, \ldots, f_N)^{\mathsf{T}} : \mathbb{R}^N_{\ge 0} \to \mathbb{R}^N$ is called a standard interference function if for all $\boldsymbol{x}, \boldsymbol{y}$ the following holds: (i) $\boldsymbol{f} > 0$ (positivity), (ii) if $\boldsymbol{x} \le \boldsymbol{y}$ then $\boldsymbol{f}(\boldsymbol{x}) \le \boldsymbol{f}(\boldsymbol{y})$ (monotonicity), (iii) $\kappa \boldsymbol{f}(\boldsymbol{x}) > \boldsymbol{f}(\kappa \boldsymbol{x})$ for all $\kappa > 1$ (scalability). If the iteration $\boldsymbol{x}^{(n)} = \boldsymbol{f}\left(\boldsymbol{x}^{(n-1)}\right)$, $n \in \mathbb{N}$, has a fixed point $\boldsymbol{x}^*$, it is unique and the iteration converges to this fixed point from any starting vector $\boldsymbol{x}^{(0)}$. Further, if $\boldsymbol{x}^{(0)} \ge \boldsymbol{f}\left(\boldsymbol{x}^{(0)}\right)$ then convergence is componentwise monotonically decreasing to a unique fixed point.

As shown in [7], the function

$$f_i^{\boldsymbol{\Psi}}(\boldsymbol{p}) = \gamma_i \frac{p_i}{\mathrm{SINR}_i^{\boldsymbol{\Psi}}(\boldsymbol{p})} \tag{19}$$

with $\mathrm{SINR}_i^{\boldsymbol{\Psi}}(\boldsymbol{p})$ from (15) is a standard interference function provided $\psi_{ii}, \gamma_i, \sigma_i > 0$ for all $i = 1, \ldots, N$. It hence follows that for both the MF and the SIC receiver the functions $\boldsymbol{f}^{\mathrm{MF}}$ and $\boldsymbol{f}^{\mathrm{SIC}}$ derived from (19) are standard interference functions. The according iterations $\boldsymbol{p}^{(n)} = \boldsymbol{f}(\boldsymbol{p}^{(n-1)})$ converge to the fixed point $\boldsymbol{p}^*$ with componentwise minimal power, provided the requirement vector $\boldsymbol{\gamma}$ is feasible. Furthermore, the fixed point $\boldsymbol{p}^*$ satisfies $\mathrm{SINR}_i^{\boldsymbol{\Psi}}(\boldsymbol{p}^*) = \gamma_i$ for all $i = 1, \ldots, N$.

The MMSE receiver with deterministic spreading is investigated in [18]. In this work, it is shown that the power update function is a standard interference function and convergence to an optimal power assignment holds. The approximate model (14) for the MMSE receiver with random spreading is more demanding to deal with. Here we will demonstrate that alike the MF and SIC receiver a unique componentwise minimal solution is obtained by iterating a certain standard interference function. Two preparatory results are needed to arrive at the main Theorem 5.

To ease notation, we first substitute the received power $a_i p_i$ by q_i, $\boldsymbol{q} = (q_1, \dots, q_n)$, regarding the path gain a_i as constant. The SINR of user i is thus determined by

$$\mathrm{SINR}_i = \frac{q_i}{\frac{1}{L}\sum_{j\neq i} \frac{q_i q_j}{q_i + \mathrm{SINR}_i\, q_j} + \sigma^2}.$$

Proposition 3: Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_N)^\mathsf{T} > 0$. The function $\boldsymbol{f}^{\boldsymbol{\gamma}}(\boldsymbol{q})$ defined by

$$f_i^{\boldsymbol{\gamma}}(\boldsymbol{q}) = \frac{\gamma_i}{L}\sum_{j\neq i} \frac{q_i q_j}{q_i + \gamma_i q_j} + \gamma_i \sigma^2, \quad \boldsymbol{q} \geq 0, \tag{20}$$

is a standard interference function. Here, the convention $\frac{0}{0} = 0$ is used.

Proof: Positivity and scalability of $\boldsymbol{f}^{\boldsymbol{\gamma}}$ are obvious. Monotonicity follows since

$$\begin{aligned} &f_i^{\boldsymbol{\gamma}}(\boldsymbol{x}) - f_i^{\boldsymbol{\gamma}}(\boldsymbol{y}) \\ &\quad = \frac{\gamma_i}{L}\Big(\sum_{j\neq i} \frac{x_i\, y_i\,(x_j - y_j) + \gamma_i\, x_j\, y_j\,(x_i - y_i)}{(x_i + \gamma_i\, x_j)\,(y_i + \gamma_i\, y_j)}\Big) \leq 0 \end{aligned}$$

for all $0 < \boldsymbol{x} \leq \boldsymbol{y}$ and $i = 1, \dots, N$. ■

Power assignment $\boldsymbol{q} \geq 0$ is the unique fixed point of $\boldsymbol{f}^{\boldsymbol{\gamma}}$ satisfying $\boldsymbol{f}^{\boldsymbol{\gamma}}(\boldsymbol{q}) = \boldsymbol{q}$ if and only if

$$\frac{q_i}{\frac{1}{L}\sum_{j\neq i} \frac{q_i q_j}{q_i + \gamma_i q_j} + \sigma^2} = \gamma_i,\ i = 1, \dots, N. \tag{21}$$

Vector $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_N)^\mathsf{T}$ is hence a fixed point of the system of equations (14) with $q_i = a_i p_i$. By the uniqueness of this fixed point, $\mathrm{SINR}_i = \gamma_i$ holds for all $i = 1, \dots, N$.

We call $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_N)^\mathsf{T}$ feasible if there exists some $\boldsymbol{q} \geq 0$ such that (21) holds.

In the following we demonstrate that for some feasible $\boldsymbol{\gamma}^{(1)}$ any componentwise smaller $\boldsymbol{\gamma}^{(0)}$ is also feasible with a reduced power allocation.

Proposition 4: Let $\boldsymbol{\gamma}^{(1)}$ be feasible with power allocation $\boldsymbol{q}^{(1)}$. If $0 \leq \boldsymbol{\gamma}^{(0)} \leq \boldsymbol{\gamma}^{(1)}$ then $\boldsymbol{\gamma}^{(0)}$ is also feasible with power allocation $\boldsymbol{q}^{(0)}$ satisfying $\boldsymbol{q}^{(0)} \leq \boldsymbol{q}^{(1)}$. If, furthermore, $\boldsymbol{\gamma}^{(0)} \neq \boldsymbol{\gamma}^{(1)}$ then $\boldsymbol{q}^{(0)} \neq \boldsymbol{q}^{(1)}$.

Proof: It holds that

$$\boldsymbol{f}_i^{\boldsymbol{\gamma}^{(0)}}\big(\boldsymbol{q}^{(1)}\big) \leq \boldsymbol{f}_i^{\boldsymbol{\gamma}^{(1)}}\big(\boldsymbol{q}^{(1)}\big). \tag{22}$$

This follows since the difference may be written as

$$\begin{aligned} &\boldsymbol{f}_i^{\boldsymbol{\gamma}^{(0)}}\big(\boldsymbol{q}^{(1)}\big) - \boldsymbol{f}_i^{\boldsymbol{\gamma}^{(1)}}\big(\boldsymbol{q}^{(1)}\big) = \big(\gamma_i^{(0)} - \gamma_i^{(1)}\big) \cdot \\ &\quad \Big(\sum_{j\neq i} \frac{{q_i^{(1)}}^2 q_j^{(1)}}{L\big(q_i^{(1)} + \gamma_i^{(0)} q_j^{(1)}\big)\big(q_i^{(1)} + \gamma_i^{(1)} q_j^{(1)}\big)} + \sigma^2\Big) \leq 0 \end{aligned}$$

Moreover, the inequality is strict if $\gamma_i^{(0)} < \gamma_i^{(1)}$.

Now define the recursion

$$\tilde{\boldsymbol{q}}^{(0)} = \boldsymbol{q}^{(1)},\ \tilde{\boldsymbol{q}}^{(n)} = \boldsymbol{f}^{\boldsymbol{\gamma}^{(0)}}\big(\tilde{\boldsymbol{q}}^{(n-1)}\big).$$

Since

$$\tilde{\boldsymbol{q}}^{(0)} = \boldsymbol{q}^{(1)} = \boldsymbol{f}^{\boldsymbol{\gamma}^{(1)}}\big(\boldsymbol{q}^{(1)}\big) \geq \boldsymbol{f}^{\boldsymbol{\gamma}^{(0)}}\big(\boldsymbol{q}^{(1)}\big) = \tilde{\boldsymbol{q}}^{(1)}, \tag{23}$$

the sequence $\tilde{\boldsymbol{q}}^{(n)}$ is monotonically decreasing and converges to a unique fixed point $\tilde{\boldsymbol{q}}^*$. From (21) we conclude that $\boldsymbol{\gamma}^{(0)}$ is feasible with power allocation $\boldsymbol{q}^{(0)} = \tilde{\boldsymbol{q}}^*$. Further, (23) and monotonicity show that $\boldsymbol{q}^{(1)} \geq \boldsymbol{q}^{(0)}$ with strict inequality whenever $\boldsymbol{\gamma}^{(0)} \neq \boldsymbol{\gamma}^{(1)}$. ■

For receivers with SINR of type (15) the componentwise minimal power allocation satisfies $\mathrm{SINR}_i^{\boldsymbol{\Psi}}(\boldsymbol{p}) = \gamma_i$ for all $i = 1, \dots, N$. By Proposition 4 this property now carries over to the MMSE receiver. Let $\mathrm{SINR}_i(\boldsymbol{q})$ denote the signal-to-interference-plus-noise ratio of user i for the MMSE receiver with power allocation $\boldsymbol{q}$, obtained as the unique fixed point of equation (14). The proof of the following Theorem is an easy consequence of Proposition 4 and the fact that $\boldsymbol{f}^{\boldsymbol{\gamma}}$ is a standard interference function.

Theorem 5: If there is a power allocation $\boldsymbol{q}$ satisfying $\mathrm{SINR}_i(\boldsymbol{q}) \geq \gamma_i$ for all $i = 1, \dots, N$, then there exists a unique componentwise minimal $\boldsymbol{q}^*$ such that

$$\mathrm{SINR}_i(\boldsymbol{q}^*) = \gamma_i \text{ for all } i = 1, \dots, N.$$

Further, $\boldsymbol{q}^*$ is obtained as the limit of the iteration $\boldsymbol{q}^{(n)} = \boldsymbol{f}^{\boldsymbol{\gamma}}\big(\boldsymbol{q}^{(n-1)}\big)$ with $\boldsymbol{f}^{\boldsymbol{\gamma}}$ defined by (20).

V. The SINR for SIC Receivers with General Random Spreading

The remaining part of this paper is dedicated to the proof of a generalization of Theorem 1. Instead of binary random spreading we allow for more general spreading distributions. The following theorem describes the SINR of a SIC receiver in this situation.

Theorem 6: Assume random spreading sequences $\boldsymbol{s}_1, \dots, \boldsymbol{s}_N$ with independent identically distributed components s_{ji}, $\mathrm{E}(s_{ji}) = 0$ and $\mathrm{E}(\boldsymbol{s}_i^\mathsf{T}\boldsymbol{s}_i) = 1$ for all i, j. Furthermore, let $\mathrm{E}(s_{ji}^4) < \infty$ and $\mathrm{E}(s_{ji}^6) < \infty$. For a successive interference cancellation receiver the SINR of user $i = 1, \dots, N$ is given by

$$\mathrm{SINR}_i = \frac{\alpha_{ii}\, a_i\, p_i}{\sum_{j\neq i} \alpha_{ij}\, a_j\, p_j + \beta_i\, \sigma^2}, \tag{24}$$

where $\alpha_{ij} = \mathrm{E}\big(({\boldsymbol{c}_i^{\mathrm{SIC}}}^\mathsf{T}\boldsymbol{s}_j)^2\big)$ and $\beta_i = \mathrm{E}\big({\boldsymbol{c}_i^{\mathrm{SIC}}}^\mathsf{T}\boldsymbol{c}_i^{\mathrm{SIC}}\big)$, $i, j = 1, \dots, N$ are given by

$$\beta_i = \left(1 - \frac{1}{L} - \frac{1}{L^2} + \mathrm{E}\left(s_{1i}^4\right)\right)^{i-1},$$

and for $i < j$,

$$\alpha_{ij} = \frac{1}{L}\left(1 - \frac{1}{L} - \frac{1}{L^2} + \mathrm{E}\left(s_{1i}^4\right)\right)^{i-1},$$

for $i = j$,

$$\begin{aligned} \alpha_{ij} = &\left(1 - \frac{2}{L} + \frac{2}{L^2}\right)^{i-1}\left(1 - \frac{1}{L}\right) + \\ &\left(1 - \frac{1}{L} - \frac{1}{L^2} + \mathrm{E}\left(s_{1i}^4\right)\right)^{i-1}\left(\frac{1}{L} - \frac{1}{L^2} + \mathrm{E}\left(s_{1i}^4\right)\right) + \\ &\left(1 - \frac{2}{L} - \frac{1}{L^2} + \mathrm{E}\left(s_{1i}^4\right)\right)^{i-1} \\ &\quad\left(-\frac{1}{L} + \frac{1}{L^2} - \mathrm{E}\left(s_{1i}^4\right) + L\,\mathrm{E}\left(s_{1i}^4\right)\right), \end{aligned}$$

and for $i > j$

$$\alpha_{ij} = \frac{1}{L}\left(1-\frac{1}{L}-\frac{1}{L^2}+\mathrm{E}\left(s_{1i}^4\right)\right)^{i-j-1} \Bigg[\left(1-\frac{2}{L}+\frac{2}{L^2}\right)^{j-1} \left(-1-\frac{1}{L}+\frac{2}{L^2}+(2L-2)\,\mathrm{E}(s_{1j}^4)\right)+ \left(1-\frac{1}{L}-\frac{1}{L^2}+\mathrm{E}\left(s_{1i}^4\right)\right)^{j-1} \left(1-\frac{1}{L}-\frac{1}{L^2}+\frac{2}{L^3}+\left(1-\frac{3}{L}\right)\mathrm{E}(s_{1j}^4)+\mathrm{E}(s_{1j}^6)\right)+ \left(1-\frac{2}{L}-\frac{1}{L^2}+\mathrm{E}\left(s_{1i}^4\right)\right)^{j-1} \left(\frac{1}{L}+\frac{1}{L^2}-\frac{2}{L^3}-\left(L+2-\frac{3}{L}\right)\mathrm{E}(s_{1j}^4) + (L-1)\,\mathrm{E}(s_{1j}^6)\right)\Bigg].$$

The proof consists of computing $\mathrm{E}\left(({\boldsymbol c_i^{\mathrm{SIC}}}^{\mathsf T}\boldsymbol s_j)^2\right)$ and $\mathrm{E}\left({\boldsymbol c_i^{\mathrm{SIC}}}^{\mathsf T}\boldsymbol c_i^{\mathrm{SIC}}\right)$. A number of elaborate intermediate steps is needed to arrive at the representations above which will be detailed in this section. Further details on this proof can be found in [19].

To ease the notation define $\boldsymbol B_{\boldsymbol x} = \boldsymbol I - \boldsymbol x\,\boldsymbol x^{\mathsf T}$ for any vector $\boldsymbol x$ and $\boldsymbol B_j = \boldsymbol B_{\boldsymbol s_j}$ for all $j \in \{1,\dots,N\}$. Hence,

$$\boldsymbol c_i^{\mathrm{SIC}} = \prod_{j=1}^{i-1}\boldsymbol B_j\,\boldsymbol s_i, \text{ and thus, } {\boldsymbol c_i^{\mathrm{SIC}}}^{\mathsf T} = \boldsymbol s_i^{\mathsf T}\prod_{j=1}^{i-1}\boldsymbol B_{i-j}, \quad (25)$$

as $\boldsymbol B_{\boldsymbol x}$ is symmetric. Further, let $\mathrm{Diag}(\boldsymbol A)$ of a square matrix $\boldsymbol A = (a_{ij})_{1\le i,j\le L}$ denote the matrix consisting of the diagonal elements of $\boldsymbol A$ on its diagonal and zeros everywhere else, i.e., $\mathrm{Diag}(\boldsymbol A) = \mathrm{diag}(a_{11}, a_{22}, \dots, a_{LL})$.

Lemma 7: Let $\boldsymbol A = (a_{jk})_{1\le j,k\le L} \in \mathbb R^{L\times L}$ be a constant symmetric $L\times L$ matrix and let $\boldsymbol x$ be a random vector with values in $\mathbb R^L$. The entries x_i, $i = 1,\dots,L$, of $\boldsymbol x$ are assumed to be stochastically independent and identically distributed with zero mean, $\mathrm{E}(x_i^2) = \frac{1}{L}$ and $\mathrm{E}(x_i^4) < \infty$. It follows

$$\mathrm{E}\left(\boldsymbol B_{\boldsymbol x}\,\boldsymbol A\,\boldsymbol B_{\boldsymbol x}\right) = \lambda_1\boldsymbol A + \lambda_2\,\mathrm{tr}\left(\boldsymbol A\right)\,\boldsymbol I + \lambda_3\,\mathrm{Diag}\left(\boldsymbol A\right).$$

with

$$\lambda_1 = 1-\frac{2}{L}+\frac{2}{L^2},\ \lambda_2 = \frac{1}{L^2} \text{ and } \lambda_3 = \mathrm{E}\left(x_1^4\right)-\frac{3}{L^2}. \quad (26)$$

Proof: We obtain

$$\begin{aligned}\mathrm{E}\left(\boldsymbol B_{\boldsymbol x}\,\boldsymbol A\,\boldsymbol B_{\boldsymbol x}\right) &= \mathrm{E}\left(\left(\boldsymbol I - \boldsymbol x\,\boldsymbol x^{\mathsf T}\right)\,\boldsymbol A\,\left(\boldsymbol I - \boldsymbol x\,\boldsymbol x^{\mathsf T}\right)\right)\\ &= \boldsymbol A - \mathrm{E}\left(\boldsymbol x\,\boldsymbol x^{\mathsf T}\right)\,\boldsymbol A - \boldsymbol A\,\mathrm{E}\left(\boldsymbol x\,\boldsymbol x^{\mathsf T}\right) + \mathrm{E}\left(\boldsymbol x\,\boldsymbol x^{\mathsf T}\,\boldsymbol A\,\boldsymbol x\,\boldsymbol x^{\mathsf T}\right)\\ &= \boldsymbol A - 2\,\mathrm{E}\left(\boldsymbol x\,\boldsymbol x^{\mathsf T}\right)\,\boldsymbol A + \mathrm{E}\left(\boldsymbol x\,\boldsymbol x^{\mathsf T}\,\boldsymbol A\,\boldsymbol x\,\boldsymbol x^{\mathsf T}\right) \quad (27)\end{aligned}$$

as $\boldsymbol A$ and $\boldsymbol x\,\boldsymbol x^{\mathsf T}$ are symmetric.

For the different terms in (27) we obtain, $\mathrm{E}\left(\boldsymbol x\,\boldsymbol x^{\mathsf T}\right) = \mathrm{E}(x_1^2)\,\boldsymbol I = \frac{1}{L}\boldsymbol I$ and

$$\mathrm{E}\left(\left(\boldsymbol x\,\boldsymbol x^{\mathsf T}\,\boldsymbol A\,\boldsymbol x\,\boldsymbol x^{\mathsf T}\right)_{jk}\right) = \mathrm{E}\left(\sum_{l=1}^{L}\sum_{m=1}^{L} x_j\,x_k\,x_l\,x_m\,a_{lm}\right). \quad (28)$$

As $\mathrm{E}\left(x_j\right) = 0$ and due to stochastic independence all terms in (28) vanish, except for those where either $j \ne k$, $j = l$ and $k = m$ or $j \ne k$, $j = m$ and $k = l$ or $j = k$ and $l = m$, therefore,

$$\begin{aligned}\mathrm{E}\left(\left(\boldsymbol x\,\boldsymbol x^{\mathsf T}\,\boldsymbol A\,\boldsymbol x\,\boldsymbol x^{\mathsf T}\right)_{jk}\right) &= \begin{cases}\mathrm{E}\left(x_j^4\right)\,a_{jj} + \mathrm{E}\left(x_j^2\right)\sum_{l\ne j}\mathrm{E}\left(x_l^2\right)a_{ll}, & \text{for all } j = k,\\ 2\,\mathrm{E}\left(x_j^2\right)\,\mathrm{E}\left(x_k^2\right)\,a_{jk}, & \text{otherwise.}\end{cases}\\ &= \begin{cases}\mathrm{E}\left(x_1^4\right)\,a_{jj} + \frac{1}{L^2}\sum_{l\ne j} a_{ll}, & \text{for all } j = k,\\ \frac{2}{L^2}\,a_{jk}, & \text{otherwise.}\end{cases}\end{aligned}$$

In the previous step we also used the fact that A is symmetric. Using $\boldsymbol A$, $\mathrm{tr}(\boldsymbol A)$ and $\mathrm{Diag}(\boldsymbol A)$, the above can be expressed as

$$\mathrm{E}\left(\boldsymbol x\,\boldsymbol x^{\mathsf T}\,\boldsymbol A\,\boldsymbol x\,\boldsymbol x^{\mathsf T}\right) = \frac{2}{L^2}\boldsymbol A + \frac{1}{L^2}\,\mathrm{tr}\left(\boldsymbol A\right)\,\boldsymbol I + \left(\mathrm{E}\left(x_1^4\right)-\frac{3}{L^2}\right)\mathrm{Diag}\left(\boldsymbol A\right),$$

which concludes the proof. ■

Lemma 8: Let $\boldsymbol A$, $\tilde{\boldsymbol A}$ and $\boldsymbol C$ be real valued random $L\times L$ matrices which are not necessarily independent. Further, assume $\boldsymbol A$ is symmetric. The L-dimensional random vector $\boldsymbol x$ shall be have independent and identically distributed real valued components x_i with $\mathrm{E}(x_i) = 0$, $\mathrm{E}(x_i^2) = \frac{1}{L}$ and $\mathrm{E}(x_i^4) < \infty$. Additionally, $\boldsymbol x$ shall be independent of $\boldsymbol A$, $\tilde{\boldsymbol A}$ and $\boldsymbol C$. Let

$$f(\boldsymbol A) = \lambda_1\boldsymbol A + \lambda_2\,\mathrm{tr}\left(\boldsymbol A\right)\,\boldsymbol I + \lambda_3\,\mathrm{Diag}\left(\boldsymbol A\right)$$

with λ_1, λ_2 and λ_3 as defined in (26). It holds,

$$\mathrm{E}\left(\boldsymbol C\,\boldsymbol B_{\boldsymbol x}\,\boldsymbol A\,\boldsymbol B_{\boldsymbol x}\,\boldsymbol C\,\tilde{\boldsymbol A}\right) = \mathrm{E}\left(\boldsymbol C\,f(\boldsymbol A)\,\boldsymbol C\,\tilde{\boldsymbol A}\right)$$

Proof: Denote the dimensions of matrix $\boldsymbol C\,\boldsymbol B_{\boldsymbol x}\,\boldsymbol A\,\boldsymbol B_{\boldsymbol x}\,\boldsymbol C\,\tilde{\boldsymbol A}$ by m, n. Conditioning under $\boldsymbol A, \tilde{\boldsymbol A}$ and $\boldsymbol C$ we get for each component i, j, with $1 \le i \le n$ and $1 \le j \le m$, of this matrix,

$$\begin{aligned}\mathrm{E}\left(\left(\boldsymbol C\,\boldsymbol B_{\boldsymbol x}\,\boldsymbol A\,\boldsymbol B_{\boldsymbol x}\,\boldsymbol C\,\tilde{\boldsymbol A}\right)_{i,j}\right) &= \int \mathrm{E}\left(\left(\boldsymbol C\,\boldsymbol B_{\boldsymbol x}\,\boldsymbol A\,\boldsymbol B_{\boldsymbol x}\,\boldsymbol C\,\tilde{\boldsymbol A}\right)_{i,j}\middle| \boldsymbol C = \boldsymbol C_1, \boldsymbol A = \boldsymbol A_1, \tilde{\boldsymbol A} = \tilde{\boldsymbol A}_1\right) dP^{(\boldsymbol C,\boldsymbol A,\tilde{\boldsymbol A})}(\boldsymbol C_1, \boldsymbol A_1, \tilde{\boldsymbol A}_1)\\ &= \int \mathrm{E}\left(\left(\boldsymbol C_1\,\boldsymbol B_{\boldsymbol x}\,\boldsymbol A_1\,\boldsymbol B_{\boldsymbol x}\,\boldsymbol C_1\,\tilde{\boldsymbol A}_1\right)_{i,j}\right) dP^{(\boldsymbol C,\boldsymbol A,\tilde{\boldsymbol A})}(\boldsymbol C_1, \boldsymbol A_1, \tilde{\boldsymbol A}_1)\\ &= \int \left(\boldsymbol C_1\,\mathrm{E}\left(\boldsymbol B_{\boldsymbol x}\,\boldsymbol A_1\,\boldsymbol B_{\boldsymbol x}\right)\,\boldsymbol C_1\,\tilde{\boldsymbol A}_1\right)_{i,j} dP^{(\boldsymbol C,\boldsymbol A,\tilde{\boldsymbol A})}(\boldsymbol C_1, \boldsymbol A_1, \tilde{\boldsymbol A}_1).\end{aligned}$$

Applying Lemma 7 inside the integral yields

$$\begin{aligned}\mathrm{E}\left(\left(\boldsymbol{C}\,\boldsymbol{B_x}\,\boldsymbol{A}\,\boldsymbol{B_x}\,\boldsymbol{C}\,\tilde{\boldsymbol{A}}\right)_{i,j}\right) &= \int \left(\boldsymbol{C}_1 f(\boldsymbol{A}_1)\,\boldsymbol{C}_1\,\tilde{\boldsymbol{A}}_1\right)_{i,j} dP^{(\boldsymbol{C},\boldsymbol{A},\tilde{\boldsymbol{A}})}(\boldsymbol{C}_1,\boldsymbol{A}_1,\tilde{\boldsymbol{A}}_1)\\ &= \mathrm{E}\left(\left(\boldsymbol{C}\,f(\boldsymbol{A})\,\boldsymbol{C}\,\tilde{\boldsymbol{A}}\right)_{i,j}\right)\end{aligned}$$

which proves the lemma. ■

Lemma 9: Let $f(\boldsymbol{A}) = \lambda_1\,\boldsymbol{A} + \lambda_2\,\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} + \lambda_3\,\mathrm{Diag}(\boldsymbol{A})$ for any $\boldsymbol{A} \in \mathbb{R}^{L\times L}$, $L \in \mathbb{N}$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, then

$$\begin{aligned}f^n(\boldsymbol{A}) &= \underbrace{f(f(\cdots f(\boldsymbol{A})\cdots))}_{n\text{-times}}\\ &= \lambda_1^n\,\boldsymbol{A} + \frac{(\lambda_1 + L\,\lambda_2 + \lambda_3)^n - (\lambda_1+\lambda_3)^n}{L}\,\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} +\\ &\quad (-\lambda_1^n + (\lambda_1+\lambda_3)^n)\,\mathrm{Diag}(\boldsymbol{A}).\end{aligned}$$

Furthermore, if $\boldsymbol{A}$ is symmetric, $f^n(\boldsymbol{A})$ is symmetric, too.

Proof: We will prove this lemma via complete induction. Obviously, $f^1(\boldsymbol{A}) = f(\boldsymbol{A})$. Thus, it remains to prove the inductive step. Let

$$\begin{aligned}b_1^{(n)} &= \lambda_1^n,\\ b_2^{(n)} &= \frac{(\lambda_1 + L\,\lambda_2+\lambda_3)^n - (\lambda_1+\lambda_3)^n}{L},\\ b_3^{(n)} &= -\lambda_1^n + (\lambda_1+\lambda_3)^n.\end{aligned}$$

Assume

$$f^n(\boldsymbol{A}) = b_1^{(n)}\,\boldsymbol{A} + b_2^{(n)}\,\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} + b_3^{(n)}\,\mathrm{Diag}(\boldsymbol{A})$$

holds for n. We obtain for $n+1$,

$$\begin{aligned}f^{n+1}(\boldsymbol{A}) &= f\big(f^n(\boldsymbol{A})\big)\\ &= \lambda_1\left(b_1^{(n)}\,\boldsymbol{A} + b_2^{(n)}\,\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} + b_3^{(n)}\,\mathrm{Diag}(\boldsymbol{A})\right) +\\ &\quad \lambda_2\,\mathrm{tr}\left(b_1^{(n)}\,\boldsymbol{A} + b_2^{(n)}\,\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} + b_3^{(n)}\,\mathrm{Diag}(\boldsymbol{A})\right)\boldsymbol{I} +\\ &\quad \lambda_3\,\mathrm{Diag}\left(b_1^{(n)}\boldsymbol{A} + b_2^{(n)}\,\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} + b_3^{(n)}\,\mathrm{Diag}(\boldsymbol{A})\right).\end{aligned}$$

$\mathrm{tr}(\cdot)$ and $\mathrm{Diag}(\cdot)$ are linear, therefore,

$$\begin{aligned}&f^{n+1}(\boldsymbol{A})\\ &= \lambda_1\,b_1^{(n)}\,\boldsymbol{A} + \lambda_1\,b_2^{(n)}\,\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} + \lambda_1\,b_3^{(n)}\,\mathrm{Diag}(\boldsymbol{A}) +\\ &\quad \lambda_2\,b_1^{(n)}\,\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} + \lambda_2\,b_2^{(n)}\,\underbrace{\mathrm{tr}\left(\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I}\right)}_{L\,\mathrm{tr}(\boldsymbol{A})}\boldsymbol{I} +\\ &\quad \lambda_2\,b_3^{(n)}\,\underbrace{\mathrm{tr}\left(\mathrm{Diag}(\boldsymbol{A})\right)}_{\mathrm{tr}(\boldsymbol{A})}\boldsymbol{I} + \lambda_3\,b_1^{(n)}\,\mathrm{Diag}(\boldsymbol{A}) +\\ &\quad \lambda_3\,b_2^{(n)}\,\underbrace{\mathrm{Diag}\left(\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I}\right)}_{\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I}} + \lambda_3\,b_3^{(n)}\,\underbrace{\mathrm{Diag}\left(\mathrm{Diag}(\boldsymbol{A})\right)}_{\mathrm{Diag}(\boldsymbol{A})}\\ &= \lambda_1\,b_1^{(n)}\boldsymbol{A} +\\ &\quad \left(\lambda_2\,b_1^{(n)} + (\lambda_1 + L\,\lambda_2 + \lambda_3) b_2^{(n)} + \lambda_2\,b_3^{(n)}\right)\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} +\\ &\quad \left(\lambda_3\,b_1^{(n)} + (\lambda_1+\lambda_3)\,b_3^{(n)}\right)\mathrm{Diag}(\boldsymbol{A})\end{aligned}$$

Finally, replacing $b_1^{(n)}$, $b_2^{(n)}$, and $b_3^{(n)}$ yields for the coefficients of the matrices $\boldsymbol{A}$, $\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I}$ and $\mathrm{Diag}(\boldsymbol{A})$

$$\lambda_1\,b_1^{(n)} = \lambda_1\,\lambda_1^n = \lambda_1^{n+1} = b_1^{(n+1)},$$

furthermore,

$$\begin{aligned}&\lambda_2\,b_1^{(n)} + (\lambda_1 + L\,\lambda_2+\lambda_3) b_2^{(n)} + \lambda_2\,b_3^{(n)} = \dots\\ &\qquad = \frac{(\lambda_1 + L\,\lambda_2 + \lambda_3)^{n+1} - (\lambda_1+\lambda_3)^{n+1}}{L} = b_2^{(n+1)}\end{aligned}$$

and finally for $\mathrm{Diag}(\boldsymbol{A})$

$$\lambda_3\,b_1^{(n)} + (\lambda_1+\lambda_3)\,b_3^{(n)} = -\lambda_1^{n+1} + (\lambda_1+\lambda_3)^{n+1} = b_3^{(n+1)}$$

which proves the first part of the lemma. Further, it is obvious that $f^{(n)}(\boldsymbol{A})$ is symmetric if $\boldsymbol{A}$ is symmetric, too. ■

Finally, we can prove Theorem 6.

Proof: First compute $\beta_i = \mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{c}_i^{\mathrm{SIC}}\right)$. It holds,

$$\begin{aligned}\mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{c}_i^{\mathrm{SIC}}\right) &= \mathrm{E}\left(\boldsymbol{s}_i^{\mathsf{T}}\prod_{j=1}^{i-1}\boldsymbol{B}_{i-j}\prod_{k=1}^{i-1}\boldsymbol{B}_k\,\boldsymbol{s}_i\right)\\ &= \mathrm{E}\left(\mathrm{tr}\left(\prod_{j=1}^{i-1}\boldsymbol{B}_{i-j}\prod_{k=1}^{i-1}\boldsymbol{B}_k\,\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}}\right)\right).\end{aligned}$$

Interchanging trace and expectation and applying Lemma 8 successively yields

$$\begin{aligned}\mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{c}_i^{\mathrm{SIC}}\right) &= \mathrm{tr}\left(\mathrm{E}\left(f^{(i-1)}(\boldsymbol{I})\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}}\right)\right)\\ &= \frac{1}{L}\,\mathrm{tr}\left(f^{(i-1)}(\boldsymbol{I})\right).\end{aligned}$$

Applying Lemma 9 we get after some easy algebra

$$\beta_i = \mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{c}_i^{\mathrm{SIC}}\right) = \left(1 - \frac{1}{L} - \frac{1}{L^2} + \mathrm{E}\left(s_{1i}^4\right)\right)^{i-1}.$$

For $\alpha_{ij} = \mathrm{E}\left(\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{s}_j\right)^2\right)$ we obtain for all $i,j = 1,\dots,N$,

$$\begin{aligned}\mathrm{E}\left(\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{s}_j\right)^2\right) &= \mathrm{E}\left(\boldsymbol{s}_j^{\mathsf{T}}\,\boldsymbol{c}_i^{\mathrm{SIC}}\,\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{s}_j\right)\\ &= \mathrm{E}\left(\mathrm{tr}\left(\boldsymbol{c}_i^{\mathrm{SIC}}\,\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\right)\right).\end{aligned} \qquad (29)$$

First, assume $i < j$, i.e., $\boldsymbol{c}_i^{\mathrm{SIC}}$ and $\boldsymbol{s}_j$ are stochastically independent. This yields

$$\begin{aligned}\mathrm{E}\left(\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{s}_j\right)^2\right) &= \mathrm{tr}\left(\mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}}\,\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\right)\mathrm{E}\left(\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\right)\right)\\ &= \frac{1}{L}\,\mathrm{E}\left(\mathrm{tr}\left(\boldsymbol{c}_i^{\mathrm{SIC}}\,\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\right)\right) = \frac{1}{L}\,\mathrm{E}\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{c}_i^{\mathrm{SIC}}\right)\\ &= \frac{1}{L}\left(1 - \frac{1}{L} - \frac{1}{L^2} + \mathrm{E}\left(s_{1i}^4\right)\right)^{i-1}.\end{aligned} \qquad (30)$$

Next, assume $i = j$. Starting from (29) and applying Lemma 8 successively we get

$$\begin{aligned}\mathrm{E}\left(\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{s}_i\right)^2\right) &= \mathrm{tr}\left(\mathrm{E}\left(\prod_{k=1}^{i-1}\boldsymbol{B}_k\,(\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}})\prod_{l=1}^{i-1}\boldsymbol{B}_{i-l}\,\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}}\right)\right)\\ &= \mathrm{tr}\left(\mathrm{E}\left(f^{i-1}(\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}})\,\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}}\right)\right)\\ &= \mathrm{E}\left(\boldsymbol{s}_i^{\mathsf{T}}\,f^{i-1}(\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}})\,\boldsymbol{s}_i\right).\end{aligned}$$

Exploiting Lemma 9 results in

$$\begin{aligned}\mathrm{E}\left(\left(\boldsymbol{c}_i^{\mathrm{SIC}\,\mathsf{T}}\,\boldsymbol{s}_i\right)^2\right) &= \lambda_1^{i-1}\,\mathrm{E}\left(\boldsymbol{s}_i^{\mathsf{T}}\,(\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}})\,\boldsymbol{s}_i\right) +\\ &\quad \frac{(\lambda_1 + L\,\lambda_2+\lambda_3)^{i-1} - (\lambda_1+\lambda_3)^{i-1}}{L}\,\mathrm{E}\left(\boldsymbol{s}_i^{\mathsf{T}}\,\mathrm{tr}(\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}})\,\boldsymbol{s}_i\right)\\ &\quad + \left(-\lambda_1^{i-1} + (\lambda_1+\lambda_3)^{i-1}\right)\mathrm{E}\left(\boldsymbol{s}_i^{\mathsf{T}}\,\mathrm{Diag}(\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}})\boldsymbol{s}_i\right).\end{aligned}$$

With

$$\mathrm{E}\left(\boldsymbol{s}_i^{\mathsf{T}}\,\mathrm{tr}(\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}})\,\boldsymbol{s}_i\right) = \mathrm{E}\left(\boldsymbol{s}_i^{\mathsf{T}}\,(\boldsymbol{s}_i^{\mathsf{T}}\,\boldsymbol{s}_i)\,\boldsymbol{s}_i\right) = \mathrm{E}\left(\boldsymbol{s}_i^{\mathsf{T}}\,\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}}\,\boldsymbol{s}_i\right) = 1 - \frac{1}{L} + L\,\mathrm{E}\left(s_{1i}^4\right)$$

and

$$\mathrm{E}\left(\boldsymbol{s}_i^{\mathsf{T}}\,\mathrm{Diag}(\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}})\,\boldsymbol{s}_i\right) = \mathrm{E}\left(\sum_{j=1}^{L} s_{ji}^4\right) = L\,\mathrm{E}\left(s_{1i}^4\right)$$

we obtain after some algebra for α_{ii}, $i = 1, \ldots, N$ the result stated in Theorem 6.

Finally, assume $i > j$ and start from (29). It follows,

$$\begin{aligned}
\mathrm{E}\left(\left(\boldsymbol{c}_i^{\mathrm{SIC}^{\mathsf{T}}}\,\boldsymbol{s}_j\right)^2\right) &= \mathrm{E}\left(\mathrm{tr}\left(\boldsymbol{c}_i^{\mathrm{SIC}}\,\boldsymbol{c}_i^{\mathrm{SIC}^{\mathsf{T}}}\,\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\right)\right)\\
&= \mathrm{E}\left(\mathrm{tr}\left(\prod_{k=1}^{i-1}\boldsymbol{B}_k\,\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}}\prod_{l=1}^{i-1}\boldsymbol{B}_{i-l}\,\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\right)\right)\\
&= \mathrm{E}\left(\mathrm{tr}\left(\prod_{l=1}^{i-1}\boldsymbol{B}_{i-l}\,\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\prod_{k=1}^{i-1}\boldsymbol{B}_k\,\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}}\right)\right)\\
&= \mathrm{tr}\left(\mathrm{E}\left(\prod_{l=1}^{i-1}\boldsymbol{B}_{i-l}\,\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\prod_{k=1}^{i-1}\boldsymbol{B}_k\right)\mathrm{E}\left(\boldsymbol{s}_i\,\boldsymbol{s}_i^{\mathsf{T}}\right)\right)\\
&= \frac{1}{L}\,\mathrm{tr}\left(\mathrm{E}\left(\prod_{l=1}^{i-1}\boldsymbol{B}_{i-l}\,\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\prod_{k=1}^{i-1}\boldsymbol{B}_k\right)\right),
\end{aligned}$$

where we used that the trace of a product of matrices is invariant under cyclic shifts and the stochastic independence of $\boldsymbol{s}_i\boldsymbol{s}_j^{\mathsf{T}}$ of the other random matrices. Further, if $i - 1 > j$ we get

$$\begin{aligned}
&\mathrm{E}\left(\left(\boldsymbol{c}_i^{\mathrm{SIC}^{\mathsf{T}}}\,\boldsymbol{s}_j\right)^2\right)\\
&= \frac{1}{L}\,\mathrm{E}\left(\mathrm{tr}\left(\prod_{l=2}^{i-1}\boldsymbol{B}_{i-l}\,\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\prod_{k=1}^{i-2}\boldsymbol{B}_k\,\boldsymbol{B}_{i-1}\,\boldsymbol{B}_{i-1}\right)\right)\\
&= \frac{1}{L}\,\mathrm{tr}\left(\mathrm{E}\left(\prod_{l=2}^{i-1}\boldsymbol{B}_{i-l}\,\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\prod_{k=1}^{i-2}\boldsymbol{B}_k\right)\mathrm{E}\left(\boldsymbol{B}_{i-1}\,\boldsymbol{B}_{i-1}\right)\right).
\end{aligned}$$

Using $\mathrm{E}\left(\boldsymbol{B}_{i-1}\,\boldsymbol{B}_{i-1}\right) = \left(1 - \frac{1}{L} - \frac{1}{L^2} + \mathrm{E}\left(s_{i1}^4\right)\right)\boldsymbol{I}$ according to Lemma 7, and applying the above procedure iteratively $i - j - 1$ times yields

$$\begin{aligned}
&\mathrm{E}\left(\left(\boldsymbol{c}_i^{\mathrm{SIC}^{\mathsf{T}}}\,\boldsymbol{s}_j\right)^2\right)\\
&= \frac{1}{L}\left(1 - \frac{1}{L} - \frac{1}{L^2} + \mathrm{E}\left(s_{1i}^4\right)\right)^{i-j-1}\\
&\qquad \mathrm{E}\left(\mathrm{tr}\left(\prod_{l=1}^{j}\boldsymbol{B}_{j-l+1}\,\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\prod_{k=1}^{j}\boldsymbol{B}_k\right)\right)\\
&= \frac{1}{L}\left(1 - \frac{1}{L} - \frac{1}{L^2} + \mathrm{E}\left(s_{1i}^4\right)\right)^{i-j-1}\\
&\qquad \mathrm{E}\left(\mathrm{tr}\left(\prod_{l=2}^{j}\boldsymbol{B}_{j-l+1}\,\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\prod_{k=1}^{j-1}\boldsymbol{B}_k\,\boldsymbol{B}_j\,\boldsymbol{B}_j\right)\right).
\end{aligned}$$

Applying Lemma 8 successively $j - 1$ times results in

$$\mathrm{E}\left(\left(\boldsymbol{c}_i^{\mathrm{SIC}^{\mathsf{T}}}\,\boldsymbol{s}_j\right)^2\right) = \frac{1}{L}\left(1 - \frac{1}{L} - \frac{1}{L^2} + \mathrm{E}\left(s_{1i}^4\right)\right)^{i-j-1} \mathrm{E}\left(\mathrm{tr}\left(f^{j-1}(\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}})\,\boldsymbol{B}_j\,\boldsymbol{B}_j\right)\right).$$

Utilizing Lemma 9 on the expectation yields

$$\begin{aligned}
&\mathrm{E}\left(\mathrm{tr}\left(f^{j-1}(\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}})\,\boldsymbol{B}_j\,\boldsymbol{B}_j\right)\right) =\\
&\lambda_1^n\,\mathrm{E}\left(\mathrm{tr}\left(\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\,\boldsymbol{B}_j\,\boldsymbol{B}_j\right)\right) +\\
&\frac{(\lambda_1 + L\,\lambda_2 + \lambda_3)^n - (\lambda_1 + \lambda_3)^n}{L}\mathrm{E}\left(\mathrm{tr}\left(\mathrm{tr}(\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}})\,\boldsymbol{B}_j\,\boldsymbol{B}_j\right)\right)\\
&+ \left(-\lambda_1^n + (\lambda_1 + \lambda_3)^n\right)\mathrm{E}\left(\mathrm{tr}\left(\mathrm{Diag}(\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}})\,\boldsymbol{B}_j\,\boldsymbol{B}_j\right)\right).
\end{aligned}$$

It remains to evaluate the three expectations. One obtains after some algebra for the first term

$$\mathrm{E}\left(\mathrm{tr}\left(\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}}\,\boldsymbol{B}_j\,\boldsymbol{B}_j\right)\right) = -\frac{1}{L} + \frac{2}{L^2} + (L-3)\,\mathrm{E}(s_{1j}^4) + L\,\mathrm{E}(s_{1j}^6),$$

for the second,

$$\mathrm{E}\left(\mathrm{tr}\left(\mathrm{tr}(\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}})\,\boldsymbol{B}_j\,\boldsymbol{B}_j\right)\right) = L - 1 - \frac{1}{L} + \frac{2}{L^2} + (L-3)\,\mathrm{E}(s_{1j}^4) + L\,\mathrm{E}(s_{1j}^6)$$

and finally for the third

$$\mathrm{E}\left(\mathrm{tr}\left(\mathrm{Diag}(\boldsymbol{s}_j\,\boldsymbol{s}_j^{\mathsf{T}})\,\boldsymbol{B}_j\,\boldsymbol{B}_j\right)\right) = 1 - (L+1)\,\mathrm{E}\left(s_{1j}^4\right) + L\,\mathrm{E}\left(s_{1j}^6\right).$$

Collecting the previous results we obtain after some algebra the result stated in Theorem 6 for the case $i > j$ and conclude the proof. ■

For binary random spreading it holds that $\mathrm{E}\left(s_{1i}^4\right) = \frac{1}{L^2}$ and $\mathrm{E}\left(s_{1i}^6\right) = \frac{1}{L^3}$. Applying these values in Theorem 6 proves Theorem 1.

VI. Conclusions

This paper has considered the concept of feasibility and uniformly minimal power allocation for three receiver types in CDMA radio networks. Whilst for the matched filter receiver results are well known, this paper has contributed to analyzing the linear SIC and the MMSE receiver. We have achieved an explicit form of the SINR for the former, and use an accurate large system approximation for the latter case. In both cases, the energy optimal power solution is attained at the boundary. We have also given explicit and iterative procedures to determine the minimal power solution in practice.

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