Analyzing a Distributed Slot Assignment Protocol by Markov Chains

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Abstract. Packet radio networks must be able to handle rapid changes in connectivity, e.g. for a continuous data exchange in inter–car communication. In a concurrent slot assignment protocol this is achieved by random access to time synchronized frames and exchange of individual channel observation by each of the involved stations. An important question is how fast this synchronization tool works. We investigate its performance by analyzing the (random) first stable time in a Markov chain model. Numerical results for a variety of parameters are given.

1. Introduction

We consider a set of k mobile stations which are to be connected by radio communication such that each of them is able to transmit data to each neighbour on the same frequency. Any stations with common neighbours cannot transmit data in the same slot due to superimposition. The transmission (or receiving) time is devided into frames of fixed length where overall time synchronization is assumed. Following the CSAP protocol [3] each frame consists of N slots which may be used by each of the stations to transmit blocks of data. Each slot contains a package of N bits describing the individual channel observations by CRC–checking followed by a message block.

Each station transmits "1"as the j–th bit in the channel synchronization block if its check yields an identifiable message in the j–th slot or if the j–th slot is used by the station itself and "0" otherwise. "0" may be interpreted as "slot unused" or alternatively as "signal deletion" by simultaneous transmission of two or more other stations. The channel partitioning is depicted in Fig. 1.

If a station is able to recognize its slot as error free for transmission it continues transmitting in the same slot in subsequent frames, otherwise it switches to a slot which is recognized as unused or overloaded. Correspondingly, overloaded slots will be left by stations involved and the system tends towards a stable uncollided slot assignment.

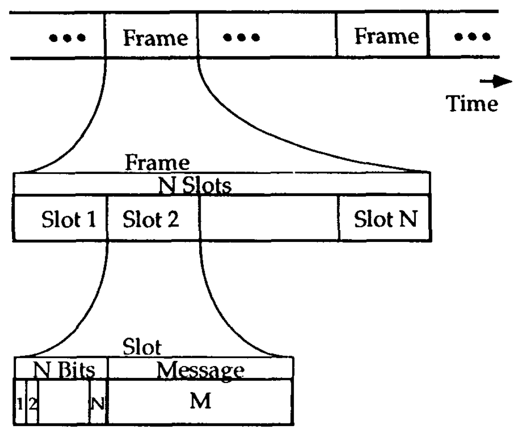

Fig. 1. Channel partitioning

An analogous strategy is applied in the DCAP protocol [2] or by R–ALOHA [1] for fully connected radio networks where each station is able to transmit data to each other directly. Consequently this procedure may be analyzed by the same methods.

In this paper, under a certain probabilistic model, we investigate how fast a stable state will be achieved. Furthermore, we consider corresponding error probabilities. Chapter 2 deals with a precise description of slot change behaviour. In Chapter 3 we introduce the probabilistic model, and conclude in Chapter 4 with theoretical analyses and reconfirm our results by numerical examples and simulations.

2. Slot changes by bit pattern

We assume a cluster of k stations such that any two stations are directly connected to each other or learn about their slot assignment by at least one common neighbour. Once a complete frame has passed by, station s is able to

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build up a bit pattern of the communication behaviour as follows. Define the binary variables b_{si}, $1 \leq i \leq N$, by

$$b_{si} = \begin{cases} 1, & \text{if station } s \text{ receives a clear signal from any other in slot } i, \\ 1, & \text{if slot } i \text{ is used by } s \text{ itself}, \\ 0, & \text{otherwise.} \end{cases}$$

$b_{si} = 0$ may be due to superimposition in slot i or absence of any HF–carrier. This individual bit pattern is transmitted by station s during the next frame in the chosen slot. If station s does not collide its bit pattern is received and evaluated by all other stations.

The same procedure is carried out by each of the stations and the corresponding information is transmitted in the communication bit block. After a complete frame, a binary $N \times N$ information matrix $C(s) = (c_{ij}(s))_{1 \leq i,j \leq N}$ is available for each station s where for $1 \leq s \leq k$

$$c_{ij}(s) = \begin{cases} b_{tj}, & \text{if slot } i \text{ is used by station } t \text{ uncollidedly, } t \neq s, \\ b_{sj}, & \text{if station } s \text{ uses slot } i \text{ itself}, \\ 0, & \text{otherwise,} \end{cases}$$

Let us assume that station s transmits in slot i. Applying the logical OR to the elements of each column of $C(s)$ in the following way

$$\bar{c}_j(s) = \mathrm{OR}\big(c_{1j}(s), \ldots, c_{i-1,j}(s), c_{i+1,j}(s), \ldots, c_{N,j}(s)\big),$$

$1 \leq j \leq N$, yields

$$\bar{c}_j(s) = \begin{cases} 1, & \text{if slot } j \text{ is used by exactly one station,} \\ 0, & \text{if slot } j \text{ is superimposed or unoccupied.} \end{cases}$$

$b_{si} = 1$ holds since station s transmits in slot i. In case that $(\bar{c}_i(s)\ \mathrm{AND}\ b_{si}) = 1$ the slot choice of station s does not lead to a collision with any other and its transmission works. The result $(\bar{c}_i(s)\ \mathrm{AND}\ b_{si}) = 0$ shows that a collision has occured for station s, and it has to change its slot in the next frame to achieve correct communicaton. Available slots may be detected by $\bar{c}_j(s) = 0$, i.e. collided or unused ones.

Consequently each station may identify all slots j which are occupied correctly by just one transmitter $(\bar{c}_j(s) = 1)$ and slots which are either unused or overloaded $(\bar{c}_j(s) = 0)$. This also holds true for the slot station s presently uses itself.

But this merely works under the assumption that at least two stations are not colliding with any other. In case that all stations except of one collide the situation shows up somewhat different.

Let us assume that station s is transmitting in slot i and the other ones $t \neq s$ are involved in collided slots $j(t)$, $j(t) \neq i$. In this case the information matrix $C(s)$ consists of $(b_{s1}, \ldots, b_{sN})$ as i–th row and all other entries are 0. This results in $(\bar{c}_1(s), \ldots, \bar{c}_N(s)) = (0, \ldots, 0)$. All other stations $t \neq s$ receive either the bit string $(b_{s1}, \ldots, b_{sN})$ as the i–th row of $C(t)$ where $b_{sj(t)} = 0$, if they are connected with station s, or otherwise $c_{ij}(t) = 0$ for all $j = 1, \ldots, N$. If $\ell \neq i$ and $\ell \neq j(t)$, $1 \leq \ell \leq N$, then, because of superimposition, $c_{\ell j(t)}(t) = 0$ holds. In fact, $(\bar{c}_1(t), \ldots, \bar{c}_N(t)) = (0, \ldots, 0)$ for all $t = 1, \ldots, k$, $t \neq s$. Then $(\bar{c}_1(s), \ldots, \bar{c}_N(s)) = (0, \ldots, 0)$ holds for all stations s, and in the next frame each of them will choose a new slot.

The case that all stations are using collided slots is even worse, but, as may be seen easily, yields the same result of a global slot change.

In summary, if less than two stations are able to transmit error free the whole group recognizes each slot as overloaded or unused, and in the next frame each station employs an arbitrary new slot for transmission. On the other hand, if two or more stations transmit without collisions then uncollidedly used slots may be detected by each station of the group. These will not be affected by future slot changes.

3. The probabilistic model

The relevant quantities to describe the behaviour of the transmission system are

- $u \in \mathbb{N}_0$: the number of unused slots,
- $v \in \mathbb{N}_0$: the number of slots occupied by exactly one station,
- $w \in \mathbb{N}_0$: the number of slots superimposed by at least two stations,

where obviously $u + v + w = N$. Under our assumption of complete reachability for a group of k stations, $k \leq N$, and a perfect channel we obtain the additional restrictions $0 \leq w \leq \lfloor k/2 \rfloor$, $0 \leq v \leq k - 2w$, and if $v < k$ then $w \geq 1$. Here $\lfloor a \rfloor$ denotes the largest integer less than or equal to $a \in \mathbb{R}$.

These conditions may be equivalently transformed to

$$0 \leq v, w, \quad v + 2w \leq k, \quad v + k(w-1) \geq 0.$$

By this, the *state space* which completely describes the slot change behaviour is

$$\hat{\mathcal{X}} = \{x = (u, v, w) \mid u, v, w \in \mathbb{N}_0,\ u + v + w = N,\ v + 2w \leq k,\ v + k(w-1) \geq 0\} \tag{1}$$

and exactly one state $\widehat{x}^* = (N-k, k, 0) \in \widehat{\mathcal{X}}$ characterizes the transmission system as stable which means that all stations make use of different slots. $\widehat{\mathcal{X}}$ is a grid in a two dimensional flat of $\mathbf{R}^3$. In this sense it is overparametrized, but the following considerations are clearer by the above notation.

The cardinality of $\widehat{\mathcal{X}}$ may be determined by denumerating all pairs of integers (v, w) which satisfy the inequalities in the definition of $\widehat{\mathcal{X}}$. It holds

$$\#(\widehat{\mathcal{X}}) = \begin{cases} \frac{k^2}{4} + 1, & \text{if } k \text{ is even,} \\ \frac{(k-1)(k+1)}{4} + 1, & \text{if } k \text{ is odd.} \end{cases} \tag{2}$$

This number increases quadratically with k but remarkably does not depend on the number N of available slots.

In the following we are interested in the number of frames which have to pass by until perfect communication is achieved. Thus, we may neglect the behaviour within colliding groups and restrict our attention to the number v of correctly occupied slots. From (1) we recognize

$$\mathcal{X} = \{0, 1, \ldots, k-2, k\} \tag{3}$$

as the relevant state space for this problem.

We assume that initially each of the k stations independently chooses any of the N slots with probability $\frac{1}{N}$. This yields a random event in the state space $\mathcal{X}$ which is modelled by a random variable

$$X : (\Omega, \mathcal{A}, P) \longrightarrow (\mathcal{X}, \mathfrak{P}(\mathcal{X})),$$

with the interpretation $X = v$ if just v slots are occupied correctly. $(\Omega, \mathcal{A}, P)$ is an appropriate probability space and $\mathcal{X}$ is endowed with $\mathfrak{P}(\mathcal{X})$, the set of all subsets of $\mathcal{X}$, to form a measurable space.

We have to investigate the distribution of X, and for this purpose introduce random variables $S_1, \ldots, S_N$ with values in $\{0, 1, \ldots, k\}$ and the interpretation

$$S_i = j : \quad \text{slot } i \text{ is used by } j \text{ stations,} \\ 1 \le i \le N,\ 0 \le j \le k,\ k \le N.$$

It is well known that the random vector $(S_1, \ldots, S_N)$ follows a multinomial distribution with

$$P(S_1 = k_1, \ldots, S_N = k_N) = \frac{k!}{k_1! \cdots k_N!} \frac{1}{N^k}, \\ k_1, \ldots k_N \in \mathbf{N}_0,\ \sum_{i=1}^N k_i = k.$$

The distribution of $(S_1, \ldots, S_N)$ is symmetric in permutations of the indices such that for all $v \in \mathcal{X}$

$$\begin{aligned} &P(X = v) \\ &= \frac{N!}{(N-v)!v!} P(S_1, \ldots, S_{N-v} \in \{0, 2, \ldots, k\}, \\ &\qquad\qquad S_{N-v+1} = 1, \ldots, S_N = 1) \\ &= \frac{N!}{(N-v)!v!} \sum_{\substack{k_1, \ldots, k_{N-v} \in \{0,2,\ldots,k\} \\ k_1 + \cdots + k_{N-v} = k-v}} \frac{k!}{k_1! \cdots k_{N-v}!} \cdot \frac{1}{N^k} \\ &= \frac{N!}{v!} \cdot \frac{k!}{N^k} \cdot \\ &\sum_{\substack{k_1 \le \cdots \le k_{N-v} \in \{0,2,\ldots,k\} \\ k_1 + \cdots + k_{N-v} = k-v}} \frac{1}{k_1! \cdots k_{N-v}!} \cdot \frac{1}{h(k_1, \ldots, k_{N-v})}, \end{aligned} \tag{4}$$

where $h(k_1, \ldots, k_{N-v})$ denotes the product of factorials of the numbers of identical indices in $k_1, \ldots, k_{N-v}$. If the set of admissible indices in the above sum happens to be empty we define its value as zero.

A special case of (4) may be of interest. If $v = k$ we have the probability that no slot is occupied twice. Evaluating the above formula we obtain $P(X = k) = \frac{N!}{(N-k)!N^k}$ which is well known from the so called "coincidence or birthday problem" (cp. [4]).

For $N, k \in \mathbf{N}$, $k \le N$, and $v \in \mathbf{N}_0$ let us generally introduce the function

$$\begin{aligned} f(N, k, v) &= \frac{N!\,k!}{v!\,N^k} \cdot \\ &\sum_{\substack{k_1 \le \cdots \le k_{N-v} \in \{0,2,\ldots,k\} \\ k_1 + \cdots + k_{N-v} = k-v}} \frac{1}{k_1! \cdots k_{N-v}!} \cdot \frac{1}{h(k_1, \ldots, k_{N-v})}, \end{aligned} \tag{5}$$

if $0 \le v \le k \le N$, and $f(N, k, v) = 0$, otherwise.

If k and N are fixed f represents a discrete density function with support $\mathbf{N}_0$ which coincides with (4) on $\mathcal{X}$.

There is no basic problem to write a computer subroutine which calculates the function $f(N, k, v)$ for admissible parameters $N, k \in \mathbf{N}$, $v \in \mathbf{N}_0$. The only nonstandard part is to enumerate the summation indices in (5) efficiently. The following procedure (in a PASCAL-like notation) solves this task for `k: array[1..w] of integer` with `w:=N-v`, `r:=k-v`, and initial value `sum:=0`. A first call `nexttupel(w-1)` yields all indices in (5).

```
procedure nexttupel(j:  integer);
var i,l:  integer;
function sucsr(m:  integer):  integer;
begin
   if m=0 then sucsr:=m+2 else sucsr:=m+1;
end;
begin
   if sum+(w-j)*sucsr(k[j]) <= r-1 then begin
      l:=sucsr(k[j]); sum:=sum-k[j]+(w-j)*l;
      for i:=j to w-1 do k[i]:=l;
      k[w]:=r-sum; j:=w-1;
   end else begin
      sum:=sum-k[j]; j:=j-1;
   end;
   if j >= 1 then nexttupel(j);
end;
```

In the following we assume that $f(N,k,v)$ may be evaluated at arbitrary precision.

We now come back to the dynamic properties of the communication system. To describe the step by step behaviour a sequence of random variables $\{X_n\}_{n\in\mathbb{N}_0}$ is used each satisfying $X_n : (\Omega, \mathcal{A}, P) \rightarrow (\mathcal{X}, \mathfrak{P}(\mathcal{X}))$, $n \in \mathbb{N}_0$. X_n denotes the number of slots occupied by just one station in the n-th step.

The initial step already has been described: each station randomly chooses any slot. Thus, the initial distribution of the system is given by

$$P(X_0 = v_0) = f(N, k, v_0), \quad v_0 \in \mathcal{X}. \tag{6}$$

The state achieved in each step only depends on its direct predecessor such that the sequence $\{X_n\}_{n\in\mathbb{N}_0}$ forms a Markov chain. The transition behaviour is described in Chapter 2. In each step all colliding stations randomly choose any slot out of the superimposed or unoccupied ones (e.g. for station s these are all slots with $\bar{c}_j(s) = 0$). If there are less than two uncollided stations then all will choose a new slot. Thus, for all $v_{n+1}, v_n \in \mathcal{X}$ the corresponding transition probabilities from v_n to v_{n+1} are given by

$$P(X_{n+1} = v_{n+1} \mid X_n = v_n) = \begin{cases} f(N - v_n, k - v_n, v_{n+1} - v_n), & \text{if } v_n \geq 2, \\ f(N, k, v_{n+1}), & \text{if } v_n \leq 1, \end{cases} \tag{7}$$

with f from (5).

Hence the slot change process of a connected k-group of stations may be described by a homogeneous Markov chain with state space (3), initial distribution (6), and transition probabilities (7), which all may be calculated by evaluating the function f.

k \ N	5	10	15	20	25	30	35	40	45	50	60	70	80	90	100
5	3.70 *3.39*	1.92 *0.63*	1.61 *0.42*	1.46 *0.34*	1.37 *0.29*	1.31 *0.25*	1.27 *0.22*	1.24 *0.20*	1.21 *0.18*	1.19 *0.17*	1.16 *0.14*	1.14 *0.12*	1.12 *0.11*	1.11 *0.10*	1.10 *0.09*
10	..	4.82 *3.14*	2.77 *0.69*	2.32 *0.46*	2.10 *0.38*	1.96 *0.34*	1.86 *0.33*	1.78 *0.32*	1.71 *0.31*	1.66 *0.31*	1.57 *0.30*	1.51 *0.29*	1.46 *0.27*	1.41 *0.26*	1.38 *0.25*
15	..	..	5.62 *3.26*	3.38 *0.76*	2.83 *0.53*	2.54 *0.42*	2.36 *0.36*	2.23 *0.31*	2.14 *0.28*	2.06 *0.27*	1.95 *0.25*	1.87 *0.25*	1.80 *0.26*	1.74 *0.26*	1.70 *0.27*
20	..	..	..	6.22 *3.32*	3.85 *0.81*	3.23 *0.56*	2.90 *0.46*	2.68 *0.41*	2.52 *0.36*	2.41 *0.32*	2.24 *0.26*	2.14 *0.23*	2.06 *0.21*	2.00 *0.20*	1.94 *0.20*
25	..	..	..	..	6.68 *3.37*	4.24 *0.84*	3.57 *0.58*	3.21 *0.47*	2.97 *0.42*	2.79 *0.39*	2.54 *0.34*	2.38 *0.29*	2.27 *0.25*	2.19 *0.21*	2.13 *0.19*
30	..	..	..	..	..	7.06 *3.40*	4.57 *0.86*	3.86 *0.60*	3.47 *0.48*	3.21 *0.41*	2.88 *0.36*	2.66 *0.34*	2.50 *0.31*	2.38 *0.27*	2.29 *0.24*
35	..	..	..	..	..	..	7.39 *3.42*	4.85 *0.88*	4.12 *0.61*	3.70 *0.50*	3.23 *0.38*	2.95 *0.34*	2.75 *0.33*	2.60 *0.31*	2.48 *0.29*
40	..	..	..	..	..	..	..	7.67 *3.43*	5.10 *0.89*	4.35 *0.62*	3.62 *0.44*	3.25 *0.35*	3.01 *0.31*	2.83 *0.31*	2.69 *0.31*
45	..	..	..	..	..	..	..	..	7.93 *3.45*	5.33 *0.90*	4.11 *0.51*	3.58 *0.41*	3.26 *0.33*	3.06 *0.29*	2.90 *0.29*
50	..	..	..	..	..	..	..	..	..	8.15 *3.46*	4.74 *0.64*	3.97 *0.46*	3.55 *0.38*	3.28 *0.31*	3.10 *0.28*
60	..	..	..	..	..	..	..	..	..	..	8.54 *3.47*	5.08 *0.65*	4.27 *0.46*	3.82 *0.39*	3.53 *0.35*
70	..	..	..	..	..	..	..	..	..	..	..	8.87 *3.48*	5.37 *0.66*	4.53 *0.47*	4.06 *0.38*
80	..	..	..	..	..	..	..	..	..	..	..	..	9.16 *3.49*	5.62 *0.66*	4.76 *0.48*
90	..	..	..	..	..	..	..	..	..	..	..	..	..	9.41 *3.50*	5.85 *0.67*
100	..	..	..	..	..	..	..	..	..	..	..	..	..	..	9.64 *3.51*

Table 1. Expected values and variances (below) of Sync(N, k)

4. First step analysis

As described in Chap. 3 perfect communication works whenever $k \in \mathcal{X}$ is achieved by the system. From (7) we see that k is an absorbing state since for all $v \in \mathcal{X}$

$$P(X_{n+1} = v \mid X_n = k) = f(N-k,0,0) = 1.$$

The number $S_{N,k}$ of frames and corresponding time units passing by until the system realizes this stable state for the first time is a discrete random variable

$$S_{N,k} = \min\{n \in \mathbb{N}_0 \mid X_n = k\}. \tag{8}$$

In the following the expectation and variance of $S_{N,k}$ will be determined. For this purpose let $\Pi = (p_{ij})_{i,j \in \mathcal{X}}$ denote the transition matrix of the Markov chain $\{X_n\}_{n \in \mathbb{N}_0}$ where $p_{ij} = P(X_{n+1} = j \mid X_n = i)$, $i,j \in \mathcal{X}$ (independent of n) is determined by (7). The initial distribution is characterized by the stochastic vector $p = \big(P(X_0 = 0), \ldots, P(X_0 = k-2), P(X_0 = k)\big)$.

Let $\widetilde{p}$ and $\widetilde{\Pi}$ denote the $(k-1)$-vector and $(k-1) \times (k-1)$- matrix respectively which are obtained from p and Π by deleting the last row and column, i.e.

$$\widetilde{p} = \big(P(X_0 = 0), \ldots, P(X_0 = k-2)\big),$$
$$\widetilde{\Pi} = (p_{ij})_{0 \le i,j \le k-2}.$$

If the fundamental matrix $(I_{k-1} - \widetilde{\Pi})$ is nonsingular the expectation of $S_{N,k}$ from (8) may be calculated by (cp. [5], p.351)

$$E(S_{N,k}) = \sum_{\ell=1}^{\infty} \widetilde{p}\,\widetilde{\Pi}^{\ell}\mathbf{1}_{k-1} = \widetilde{p}\,(I_{k-1} - \widetilde{\Pi})^{-1}\mathbf{1}_{k-1} \tag{9}$$

where I_{k-1} denotes the $(k-1) \times (k-1)$-identity matrix and $\mathbf{1}_{k-1}$ the $(k-1)$-vector with all components one. From (9) we derive the expectation of the first passage time into perfect synchronization as

$$E(\mathrm{Sync}(N,k)) = E(S_{N,k}) + 1. \tag{10}$$

The variance of $\mathrm{Sync}(N,k)$ is determined by analogous methods. If $(I_{k-1} - \widetilde{\Pi})$ is nonsingular it holds that

$$\begin{aligned} V(\mathrm{Sync}(N,k)) &= E(S_{N,k}^2) - \big(E(S_{N,k})\big)^2 \\ &= \sum_{\ell=1}^{\infty}(2\ell+1)(\widetilde{p}\,\widetilde{\Pi}^{\ell}\mathbf{1}_{k-1}) - \Big(\sum_{\ell=1}^{\infty}\widetilde{p}\,\widetilde{\Pi}^{\ell}\mathbf{1}_{k-1}\Big)^2 \\ &= \widetilde{p}\Big(2\,\widetilde{\Pi}(I_{k-1} - \widetilde{\Pi})^{-2} + (I_{k-1} - \widetilde{\Pi})^{-1}\Big)\mathbf{1}_{k-1} \\ &\quad - \Big(\widetilde{p}\,(I_{k-1} - \widetilde{\Pi})^{-1}\mathbf{1}_{k-1}\Big)^2. \end{aligned} \tag{11}$$

We have carried out the extensive calculations in (10) and (11) for values of $N = 5, \ldots, 100\,(5)$, and $k = 5, \ldots, 100(5)$. The corresponding expectations and variances are listed in Table 1.

The results are very encouraging. Even if the channel is overloaded by as many stations as slots are available $(k = N)$ the increase in synchronization time is quite moderate. The corresponding values may be read off the diagonal and range from 3.7 for k=5 to 9.64 for k=100. The corresponding variances show up a similar slowly increasing tendancy. A realistic example is $N = 100$ and $k = 50$. Here we have an expected first synchronization time of merely 3.1 frames with a low variance 0.28. Comparable results were obtained by preliminary simulation studies.

A good feeling of the overall behaviour may be gained from the graphical representation in Fig. 2 where the (N,k)-values in the plane are plotted against the corresponding expected values of $\mathrm{Sync}(N,k)$ on the z-axis.

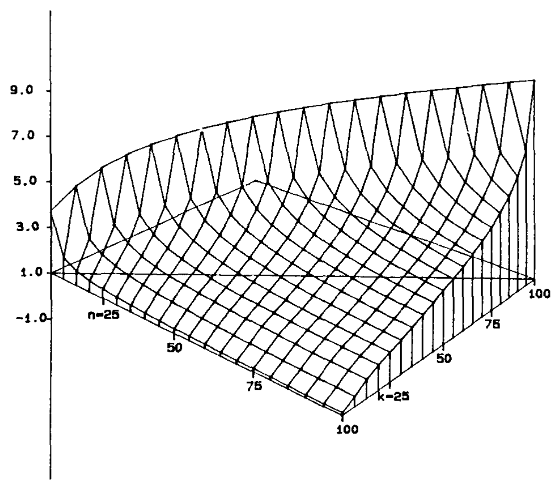

Fig. 2. Expected values of the first synchronization time

5. Conclusions

In summary, the concurrent slot assignment protocol and related systems work very well in building up noncentral communication networks. The results may serve as a worst case bound for groups which are not completely connectable. The same holds true with $k = k_1 + k_2$ when two stable groups with k_1 and k_2 stations respectively are mixing since they are moving towards each other.

Future work will be devoted to extensive simulations of the considered systems and the development of extended theoretical models for partially connected networks.

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