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COMMUNICATION THEORY

On fair rate adaption in interference-limited systems

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ABSTRACT

A fair allocation of scarce resources is crucial in systems where multiple entities compete for the same goods. General interference-limited communication systems with rate adaption are investigated in this paper and the problem of fair resource allocation is addressed by two different approaches. First, a non game theoretic fairness approach is applied to the system model. Then bargaining theory is exploited to derive a game theoretic fairness concept. To compensate the information transmission necessary in the bargaining game, so-called incentive parameters are introduced. The solution of the thereby obtained local problem coincides with the Nash bargaining solution of the global problem if the incentive parameters are properly chosen. Numerical results show the advantage of the game theoretical modelling with respect to fairness and efficiency. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

With the increasing demand for wireless communication services, better radio resource management, particularly sophisticated resource allocation, is needed. One of the first publications considering resource allocation in connection with the signal-to-interference ratio (SIR) is Ref. [1]. In an elegant setup, the author develops a general framework for proving convergence of a whole class of power assignment algorithms. Since then, various interesting approaches and algorithms on this topic have been presented. One advancement is the implementation of rate adaption in systems as this is known to increase throughput remarkably compared to fixed rate systems. In recent years, game theory has attracted attention in connection with resource management. In particular, many publications focus on noncooperative games to model power control in wireless systems. Reference [2] presents a noncooperative power control game which is shown to have a Nash equilibrium. In Ref. [3] the approach is extended to the multiple receiver case. A noncooperative game for a multirate code-division multiple access system where all users have the same rate and vary their data rate is presented in Ref. [4]. A similar approach is presented in Ref. [5] where a single cell noncooperative power control game is analysed. A good overview of activities in this field is given in Ref. [6]. One well-known deficiency of these games is that the resulting Nash equilibrium is not necessarily Pareto optimal and that and pricing concepts only provide limited remedy. That is the reason why we concentrate on cooperative game theory in this work.

In this paper we consider systems with rate adaption and approach the task of fair rate allocation from two sides. First, we investigate the fairness and efficiency of a non game theoretic max–min, sum-rate and log-utility approach. Then, we model the resource allocation as a cooperative game, a bargaining game. This axiomatic branch in game theory provides a suitable approach to address the problem as it guarantees fairness by three axioms, which are used to ensure desirable properties of the solution, and efficiency by Pareto optimality (PO). It is important to note that the Pareto optimal Nash bargaining solution (NBS) and the Nash equilibrium are different, not related concepts. In a cooperative game the performance of each player may be better than the performance achieved in a noncooperative game at a Nash equilibrium. Cooperative bargaining theory has been applied to a number of resource allocation scenarios. In Ref. [7], bargaining theory is applied to load balancing in distributed systems. In Ref. [8], a NBS is derived for a bandwidth allocation game in broadband networks with elastic traffic. OFDMA networks are considered in Ref. [9], where a NBS is discussed as a fair power allocation. Reference [10] presents a bargaining game for shared networks, where both symmetric and asymmetric bargaining models are applied. Cooperative game theory is also the topic in reference [11], where the feasible set of SIR is analysed by studying both NBS and proportional fairness. An elegant correlation to the proportionally fair resource allocation is derived for certain log-convex utility functions. Further, cooperative games have proven useful in spectrum allocation as shown in Refs. [12, 13].

This paper is organised as follows. In Section 2 we present the system model and prerequisites for the fairness analysis. Given QoS requirements and power restrictions, which rates can be supported by the system? And given the supported rate vectors, which one is the best to choose? To allow for the formulation of a cooperative game, we transform the original set of feasible rates to a convex set. Thereby, we model the rate assignment as a multi-objective convex optimisation problem. In Section 3, we introduce the fairness concepts, we apply to the rate adaption problem. Both non game theoretic and game theoretic concepts are presented. The solution to the bargaining game is given in Section 4. For the symmetric case, an explicit solution is obtained, whereas for the general case, a characterisation is derived. Further, this centralised approach is decentralised by using so-called incentives. Numerical results are presented in Section 5. We give an overview of the results and some proposals for possible extensions in Section 6.

2. SYSTEM MODEL

Consider a power-controlled interference-limited communication system with limited resources, T users and a SIR of user i given by

$$\mathrm{SIR}_i(p) = \frac{a_i p_i}{\sum_{ji} a_j p_j + \sigma}, \quad 1 \le i \le T \tag{1}$$

where $a_i \in (0, 1]$ denotes the channel coefficient of user i, $\boldsymbol{p} = (p_1, \cdots, p_T) > \mathbf{0}$ the vector of transmit powers and $\sigma > 0$ the noise coefficient. This system model applies, for example, to the uplink scenario from mobile users to the base station of a DS-CDMA system with a linear receiver structure. We assume rate adaption to be a feature of our system. Whereas the data rate of a user in a 3G network is fixed once he is admitted, rate adaption, as the name implies, adapts the users' data rates to the changing system loads. Provided that each user can obtain its minimum required data rate, we ask the question which one is the best to allocate?

As sectors can be treated as cells, we do not distinguish between sectors and cells. Starting point for the system analysis is the fraction of the energy per bit to the interference per chip $(E_b/I_0)_i = \omega/r_i \ \mathrm{SIR}_i(\boldsymbol{p})$ for user i, where $\omega > 0$ is the chip rate of the system and r_i is the data rate of user i, compare Refs. [14, 15]. Given lower bounds $\gamma_i > 0$ for this performance measure, $(E_b/I_0)_i \ge \gamma_i$, yields the inequalities

$$\frac{\omega}{\gamma_i r_i} \frac{a_i p_i}{\sum_{ji} a_j p_j + \sigma} \ge 1, \quad 1 \le i \le T. \tag{2}$$

From an energy efficient point of view, the system should use the smallest power setting which fulfills Equation (2). As the numerator is monotonically increasing in p_i and the denominator is monotonically increasing in p_j, the minimum is attained at the boundary and Equation (2) reduces to

$$\frac{\omega}{\gamma_i r_i} \frac{a_i p_i}{\sum_{j \ne i} a_j p_j + \sigma} = 1, \quad 1 \le i \le T. \tag{3}$$

There exists a unique solution to Equation (3) which is obtained after some algebra as, cf. Ref. [16],

$$p_i = \frac{\sigma}{a_i((\omega/\gamma_i r_i) + 1)\left(1 - \sum_{j=1}^{T} 1/((\omega/\gamma_j r_j) + 1)\right)}, \quad 1 \le i \le T.$$

Necessary and sufficient for p_i to be non-negative is $\sum_{j=1}^{T} [\omega/(\gamma_j r_j) + 1]^{-1} < 1$ which is assumed in the following. Further, we assume that the maximum transmit power of each user i is bounded by $p_{i,\max}$. Abbreviate $x_i = \omega/(\gamma_i r_i) + 1$. We transfer the problem of considering feasible data rate vectors $\boldsymbol{r} = (r_1, \ldots, r_T)$ to the problem of feasible vectors $\boldsymbol{x}$. A standard argument in queueing theory is to regard the reciprocal of the arrival rate as the average interarrival time. As $\omega/(\gamma_i r_i)$ is usually large, we may neglect the summand 1 in the definition of x_i. Up to a constant factor the values x_i then have the interpretation as average interarrival times. An elegant consequence is that the corresponding set of feasible x_i, defined below, is convex. The feasible rates can be easily obtained by $r_i = \omega/[\gamma_i(x_i - 1)]$. It holds that

$$\begin{aligned} p_i \le p_{i,\max} \quad &\Leftrightarrow \quad \frac{\sigma}{a_i x_i \left(1 - \sum_{j=1}^{T} \frac{1}{x_j}\right)} \le p_{i,\max} \\ &\Leftrightarrow \quad x_i \left(1 - \sum_{j=1}^{T} \frac{1}{x_j}\right) \ge \xi_i \\ &\Leftrightarrow \quad 1 - \sum_{j=1}^{T} \frac{1}{x_j} - \frac{\xi_i}{x_i} \ge 0 \end{aligned}$$

for all $1 \le i \le T$ with $\xi_i = \frac{\sigma}{a_i p_{i,\max}}$.

We assume that each user demands a minimum data rate and gives an upper bound for its data rate, thus, $r_i \in [r_{i,\min}, r_{i,\max}]$ with $r_{i,\max} > r_{i,\min} > 0$. It follows that

$$x_{i,\min} \le x_i \le x_{i,\max}$$

with $x_{i,\min} = \frac{\omega}{\gamma_i r_{i,\max}} + 1$ and $x_{i,\max} = \frac{\omega}{\gamma_i r_{i,\min}} + 1$.

Summarising the preliminaries, we define the fractional data rate region as

$$\mathcal{X} = \left\{ \boldsymbol{x} = (x_1, \ldots, x_T) \in \mathbb{R}^T | 1 - \sum_{j=1}^{T} \frac{1}{x_j} - \frac{\xi_i}{x_i} \ge 0, \right. \\ \left. x_{i,\min} \le x_i \le x_{i,\max}, \ 1 \le i \le T \right\}.$$

We assume that the parameters are chosen such that $\mathcal{X} \ne \emptyset$. Further, we assume that for each user i there exists some $\boldsymbol{x}$ with $x_{i,\max} > x_i$. Using the above notation, this transforms to $r_i > r_{i,\min}$ for all $1 \le i \le T$ for some rate vector $\boldsymbol{r}$. Thus, each user can achieve a data rate greater than the minimum required rate. In a scenario, where we cannot guarantee this, one or more users might be excluded by some algorithm until the conditions holds.

Note that the problem can analogously be defined in terms of the SIR instead of a the data rate. Starting point for this is the SIR of user i, see Equation (1), which is supposed to be above a certain threshold, say $\hat{\gamma}_i$. Further, as

$$x_i = \frac{\omega}{\gamma_i r_i} + 1 = \frac{1 + \frac{r_i}{\omega}\gamma_i}{\frac{r_i}{\omega}\gamma_i}$$

the auxiliary variable x_i can also be interpreteted in terms of γ_i.

All elements in the fractional data rate region fulfill the user and system requirements. In what follows, we will consider the task of selecting one element, which is superior to the other elements and a fair trade-off for each user.

For two vectors $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$, $\boldsymbol{y} = (y_1, \ldots, y_n)^{\mathrm{T}} \in \mathbb{R}^n$, we write $\boldsymbol{x} \le \boldsymbol{y}$ if $x_i \le y_i$ for all $1 \le i \le n$.

3. FAIRNESS CONCEPTS FOR RESOURCE ALLOCATION

There are many solution concepts to choose a reasonable element of $\mathcal{X}$. Clearly, the outstanding point should be Pareto optimal. Given a Pareto optimal element, the data rate of some user cannot be increased without decreasing the rate of another user. The question that arises is which of the infinitely many Pareto optimal points we should choose. One way is by introducing further sensible criteria. From the perspective of resource sharing, one of the natural criteria is the notion of fairness. However, there are many notions of fairness imaginable.

3.1. General fairness concepts

It is intuitive to choose an element of the fractional data rate region that maximises some utility. Maximising the sum of the rates is a desirable outcome and achieved by

$$\max_{\boldsymbol{x} \in \mathcal{X}} \sum_{i=1}^{T} -x_i.$$

The log-utility of the sum rate objective is used for the following problem

$$\max_{\boldsymbol{x} \in \mathcal{X}} \sum_{i=1}^{T} \log(-x_i).$$

A different, but also desirable outcome, especially with respect to fairness, is the maximisation of the minimal rate, i.e.

$$\max_{\boldsymbol{x} \in \mathcal{X}} \min_{1 \le i \le T} -x_i.$$

Max–min fairness is one of the commonly used notions of fairness and penalises large users. Max–min fairness corresponds to a Pareto optimal point. However, it is not easy to take into account that users might have different requirements.

3.2. Game theoretical fairness

A more sophisticated approach is the use of fairness from game theory, see Ref. [17]. In our cooperative resource allocation bargaining game the users are the players, and they have to agree upon some element of the fractional data rate region $\mathcal{X}$.

To analyse the game theoretic solution for our framework in more detail, we define the necessary game theoretic concepts. Consider some nonempty convex closed and componentwise upper bounded set $\mathcal{U} \subset \mathbb{R}^T$ and some element $\boldsymbol{u}^0 \in \mathbb{R}^T$ such that $\boldsymbol{u} \ge \boldsymbol{u}^0$ for some $\boldsymbol{u} \in \mathcal{U}$. The pair $(\mathcal{U}, \boldsymbol{u}^0)$ is called an T-person bargaining problem. The elements of $\mathcal{U}$ are called outcomes and $\boldsymbol{u}^0$ is the disagreement outcome. The interpretation of such a problem is as follows. A number of T bargainers, respectively mobile users, are faced with the problem that they have to negotiate for a fair point on the convex set $\mathcal{U}$. If no agreement can be reached by the bargainers, the disagreement utilities $(u_1^0, \ldots, u_T^0)$ will be the outcome of the game. Let B_T denote the family of all T-person bargaining problems. A bargaining solution is a function $F : B_T \to \mathbb{R}^T$ such that $F(\mathcal{U}, \boldsymbol{u}^0) \in \mathcal{U}$ for all $(\mathcal{U}, \boldsymbol{u}^0) \in B_T$. Nash suggested a solution that is based on certain axioms as given below.

Weak Pareto optimality (WPO).
$F : B_T \to \mathbb{R}^T$ is called weakly Pareto optimal, if for all $(\mathcal{U}, \boldsymbol{u}^0) \in B_T$ it holds that there exists no $\boldsymbol{u} \in \mathcal{U}$ satisfying $\boldsymbol{u} > F(\mathcal{U}, \boldsymbol{u}^0)$.

Symmetry (SYM).
$F : B_T \to \mathbb{R}^T$ is symmetric if for all $(\mathcal{U}, \boldsymbol{u}^0) \in B_T$ that are symmetric with respect to some subset $J \subseteq \{1, \ldots, n\}$ it holds that

$$F_i(\mathcal{U}, \boldsymbol{u}^0) = F_j(\mathcal{U}, \boldsymbol{u}^0) \quad \text{for all } i, j \in J.$$

A bargaining problem $(\mathcal{U}, \boldsymbol{u}^0) \in B_T$ is called symmetric with respect to some subset J, if

$$u_i^0 = u_j^0 \quad \text{and}$$
$$(u_1, \ldots, u_{i-1}, u_j, u_{i+1}, \ldots, u_{j-1}, u_i, u_{j+1}, \ldots, u_T) \in \mathcal{U}$$

for all $\boldsymbol{u} \in \mathcal{U}$ and for all $i, j \in J,\ i < j$.

Scale covariance (SCC).

$F : B_T \to \mathbb{R}^T$ is scale covariant if

$$F(\varphi(\mathcal{U}), \varphi(\boldsymbol{u}^0)) = \varphi(F(\mathcal{U}), (\boldsymbol{u}^0))$$

for all $\varphi : \mathbb{R}^T \to \mathbb{R}^T, \boldsymbol{u} \mapsto \tilde{\boldsymbol{u}}$ with $\tilde{u}_i = a_i u_i + b_i$ for some $a_i, b_i \in \mathbb{R}$ with $a_i > 0$ for all $1 \leq i \leq n$.

Independence of irrelevant alternatives (IIA).

$F : B_T \to \mathbb{R}^T$ is independent of irrelevant alternatives, if $F(\mathcal{U}, \boldsymbol{u}^0) = F(\tilde{\mathcal{U}}, \tilde{\boldsymbol{u}}^0)$ for all $(\mathcal{U}, \boldsymbol{u}^0)$, $(\tilde{\mathcal{U}}, \tilde{\boldsymbol{u}}^0) \in B_T$ with $\boldsymbol{u}^0 = \tilde{\boldsymbol{u}}^0$, $\mathcal{U} \subseteq \tilde{\mathcal{U}}$ and $F(\tilde{\mathcal{U}}, \tilde{\boldsymbol{u}}^0) \in \mathcal{U}$.

The interpretation of WPO is that it is impossible for all bargainers to gain with respect to the solution outcome. Symmetry, SCC and IIA are the so-called axioms of fairness. The symmetry property states that the solution does not depend on a specific label, i.e. users with both the same initial points and objectives will obtain the same performance. SCC requires the solutions to be covariant under positive affine transformations. This implies that the solution is independent of any risk neutral utility specification, see Ref. [18]. IIA demands that the solution outcome does not change when the set of possible outcomes shrinks but still contains the original solution.

A function $N : B_T \to \mathbb{R}^T$ is said to be a NBS if it holds for all $N(\mathcal{U}, \boldsymbol{u}^0) \in B_T$ that

$$N(\mathcal{U}, \boldsymbol{u}^0) = \operatorname{argmax} \left\{ \prod_{1 \leq j \leq n,\, u_j \neq u_j^0} (u_j - u_j^0) \,\middle|\, \boldsymbol{u} = (u_1, \ldots, u_T) \in \mathcal{U}, \boldsymbol{u} \geq \boldsymbol{u}^0 \right\}$$

if $\mathcal{U} \backslash \{\boldsymbol{u}^0\} \neq \emptyset$ and that $N(\mathcal{U}, \boldsymbol{u}^0) = \boldsymbol{u}^0$, if $\mathcal{U} \backslash \{\boldsymbol{u}^0\} = \emptyset$.

The NBS $\boldsymbol{u}^* = N(\mathcal{U}, \boldsymbol{u}^0)$ calls for the maximisation of the product of the users' gain from cooperation. It is well defined, as the maximum is attained at a single value. In addition it is uniquely characterised by the four axioms stated above. The only solution satisfying WPO, SYM, SCC and IIA is the NBS. A valuable attribute of the NBS is that is satisfies another axiom, namely PO.

Pareto optimality.

$F : B_T \to \mathbb{R}^T$ is called Pareto optimal, if for all $(\mathcal{U}, \boldsymbol{u}^0) \in B_T$ it holds that:

$$\boldsymbol{u} \in \mathcal{U}, \boldsymbol{u} \geq F(\mathcal{U}, \boldsymbol{u}^0) \quad \text{implies} \quad \boldsymbol{u} = F(\mathcal{U}, \boldsymbol{u}^0).$$

The interpretation of PO is that it is impossible to increase anyone's utility without decreasing the utility of some other player.

The main drawback in the NBS is that each user only considers its individual gain and does not care about how much the other users give up. In particular, Kalai and Smorodinsky argued that one's gain should be proportional to its maximum gain [19], which the NBS fails to satisfy. They proposed to modify axiom (IIA) by another axiom.

Individual monotonicity.

$F : B_T \to \mathbb{R}^T$ is called individual monotone, if, if for all $(\mathcal{U}, \boldsymbol{u}^0)$, $(\tilde{\mathcal{U}}, \boldsymbol{u}^0) \in B_T, \tilde{\mathcal{U}} \in \mathcal{U}$ with $\max_{\boldsymbol{u} \in \tilde{\mathcal{U}}} u_i = \max_{\boldsymbol{u} \in \mathcal{U}} u_i$, $\max_{\boldsymbol{u} \in \tilde{\mathcal{U}}} u_j \leq \max_{\boldsymbol{u} \in \mathcal{U}} u_j$, $i \neq j$ it holds that: $F_j(\mathcal{U}, \boldsymbol{u}^0) \geq F_j(\tilde{\mathcal{U}}, \boldsymbol{u}^0)$.

Thus, for any subset of $\mathcal{U}$ the solution for some player j cannot be improved if the maximum utility of some player i is constant.

The third well-known solution, the Thomson solution is also known as utilitarian rule. As it violates axiom SCC, the Nash and the Raiffa bargaining solution (RBS) are the more natural solutions for the rate adaption game and we will focus on those two in the following.

To embed the rate problem into the game-theoretic framework, it is important to notice that $\mathcal{X}$ is convex and closed. This follows from the fact that $-1 + \sum_{j=1}^{T} (1/x_j) + (\xi_i/x_i)$, $1 \leq i \leq T$, are convex functions on $\mathbb{R}^T_{>0}$. Further, the functions $-(x_i - x_{i,\min})$ and $-(x_{i,\max} - x_i)$ are convex. By [Ref. [20], Corollary 4.6.1] the level sets

$$\left\{ x \in \mathbb{R}^T_+ | 1 - \sum_{j=1}^{T} \frac{1}{x_j} - \frac{\xi_i}{x_i} \geq 0 \right\},$$
$$\left\{ x \in \mathbb{R}^T \,\middle|\, x_i - x_{i,\min} \geq 0 \right\},$$
$$\left\{ x \in \mathbb{R}^T \,\middle|\, x_{i,\max} - x_i \geq 0 \right\}$$

are convex for every $i \in \{1, \ldots, T\}$. Since the intersection of an arbitrary collection of convex sets is convex, cf. [Ref. [20], Theorem 2.1], we obtain the convexity of $\mathcal{X}$. Clearly, $\mathcal{X}$ is closed.

It is natural to assume that each user aims at obtaining a data rate greater than its minimum rate and as close to its maximum value as possible. When the data rate r_i of some user i tends to its maximum, then $-x_i$ tends to its maximum, too. If the users fail to reach an agreement, they end up with the minimum data rate, which is expressed by the disagreement outcome

$$\boldsymbol{u}^0 = -x_{1,\max}, \ \ldots, \ -x_{T,\max}.$$

This holds since $-x_i \geq -x_{i,\max}$ transforms to $r_i \geq r_{i,\min}$. Thus, by choosing the disagreement outcome as above, the minimum data rate for each user is assured, as desired.The notation of our game is now as follows:

$$\mathcal{U} = \mathcal{X}, \qquad \boldsymbol{u}^0 = (-x_{1,\max}, \ldots, -x_{T,\max}). \tag{4}$$

The thereby defined game has a unique NBS which is characterised as follows.

Proposition 1 The unique NBS and RBS to the bargaining problem $(\mathcal{U}, \boldsymbol{u}^0)$ defined in Equation (4) are the solutions to the following convex optimisation problem (P)

$$\begin{aligned} \text{maximize} \quad & \prod_{i=1}^{T}(x_{i,\max}-x_i)+\tfrac{\beta}{N-1}\sum_{j\neq i}(x_j-x_{j,\min}) \\ \text{subject to} \quad & 1-\sum_{j=1}^{T}\tfrac{1}{x_j}-\tfrac{\xi_i}{x_i}\geq 0,\ 1\leq i\leq T \\ & x_i-x_{i,\min}\geq 0,\ 1\leq i\leq T \\ & x_{i,\max}-x_i\geq 0,\ 1\leq i\leq T \end{aligned}$$

where $\beta=0$ for the NBS and $\beta=1$ for the RBS.

Hence, the NBS and the RBS to a bargaining problem can be calculated by implementing problem (P). Note that the feasible region in (P) is convex as shown above.

3.3. Willingness to pay

The above bargaining concepts treats all users the same with respect to their bargaining power. In reality, the bargaining result will be influenced by a variety of factors such as the tactics, the negotiation procedure or the willingness to pay of each user. Here, we assume that each user i, $1\leq i\leq T$ chooses some price $\eta_i\geq 0$ which reflects his willingness to pay, see Ref. [21]. We define the bargaining ratio of user i as $\hat{\eta}=\eta_i/\sum_{j=1}^{T}\eta_i$. The corresponding, so called asymmetric solutions are obtained by including the bargaining ratios as powers in the objective functions, i.e. the objective in (P) is replaced by

$$\prod_{i=1}^{T}\left((x_{i,\max}-x_i)+\frac{\beta}{N-1}\sum_{j\neq i}(x_j-x_{j,\min})\right)^{\hat{\eta}_i}.$$

When all users propose the same price this asymmetric models corresponds to the symmetric bargaining outcome.

4. SOLVING THE GAME

In this section, we show that a concrete solution is obtained in a special case, while for the general case we give a solution method. Further, we present a decentralisation for the centralised approach.

4.1. The symmetric case

Consider the minimum and maximum data rates of the users, and the constants ξ_i to be the same for all users, i.e. it holds that $x_{i,\min}=x_{\min}$, $x_{i,\max}=x_{\max}$ and $\xi_i=\xi$ for some $x_{\max}$, $x_{\min}$, ξ for all $1\leq i\leq T$. Then a solution to the above maximisation problem can be obtained explicitly.

Recall that the solution to (P) is a NBS, and thus satisfies the axiom SYM. Due to this axioms, the solution $\boldsymbol{x}^*$ is the same for each user, i.e. $\boldsymbol{x}^*=(x^*,\ldots,x^*)$. As the objective function is monotonically decreasing in x_i, the solution is either obtained at $x^*=x_{\min}$, or at $1-\sum_{j=1}^{T}(1/x^*)-(\xi/x^*)=0$, which is equivalent to $x^*=T+\xi$.

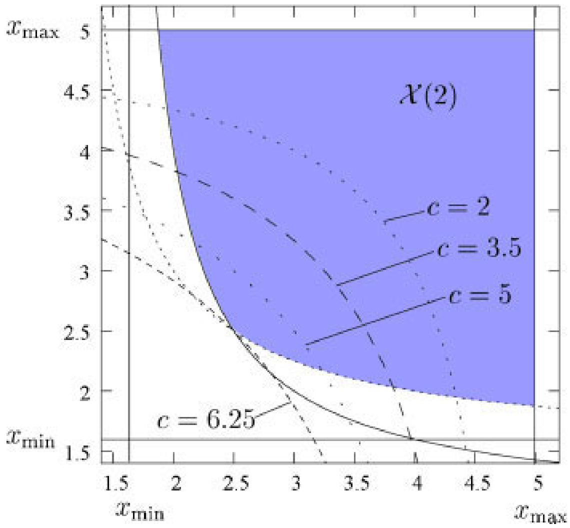

Figure 1. Region $\mathcal{X}^{(2)}$ and some level set.

Summarising the above arguments, the solution to (P) in the symmetric case is

$$\boldsymbol{x}^*=(x^*,\ldots,x^*)\quad\text{with}\quad x^*=\max\{x_{\min},T+\xi\}.$$

We consider a two- and a three-dimensional example. First, consider the two user case with $\beta=0$, $x_{\min}=1.6$, $x_{\max}=5$ and $\xi=0.5$ The corresponding fractional data rate region is

$$\mathcal{X}^{(2)}=\left\{\boldsymbol{x}\in\mathbb{R}^2\,|\,1-\sum_{j=1}^{2}\frac{1}{x_j}-\frac{0.5}{x_i}\geq 0,\ 1.6\leq x_i\leq 5,\ i\in\{1,2\}\right\}.$$

The solution to (P) is $\boldsymbol{x}^*=(2.5,2.5)$ with optimal value 6.25. Region $\mathcal{X}^{(2)}$ is depicted in Figure 1 with the level sets $(5-x_1)(5-x_2)=c$ for $c\in\{2,3.5,5,6.25\}$. The corresponding region of rates, called $\mathcal{R}^{(2)}$, is depicted in Figure 2. The corresponding parameters are $\omega=129.6$, $\gamma_i=1.5$, $r_{i,\min}=21.6$ and $r_{i,\max}=144$, $i=1,2$. The optimal rates are $r_1=r_2=57.6$. The level sets $(r_1-21.6)(r_2-21.6)=c$ are depicted for $c\in\{600,900,1296\}$. As expected, the fractional data rate region turns out to be convex, whereas the region of feasible rates is not. Second, consider the three user case with $\beta=0$, $x_{\min}=3$, $x_{\max}=5$ and $\xi=0.5$. The corresponding fractional data rate region is

$$\mathcal{X}^3=\left\{\boldsymbol{x}\in\mathbb{R}^3\,|\,1-\sum_{j=1}^{3}\frac{1}{x_j}-\frac{0.5}{x_i}\geq 0,\ 3\leq x_i\leq 5,\ i\in\{1,2,3\}\right\}.$$

The solution to (P) is thus $\boldsymbol{x}^*=(3.5,3.5,3.5)$ with optimal value 3.375. Region $\mathcal{X}^3$ is depicted in Figure 3 with the level set $(5-x_1)(5-x_2)(5-x_3)=3.375$.

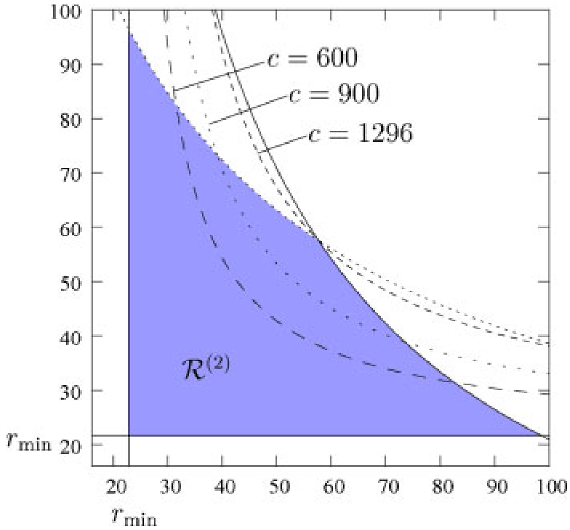

Figure 2. The corresponding region of rates, $\mathcal{R}^{(2)}$.

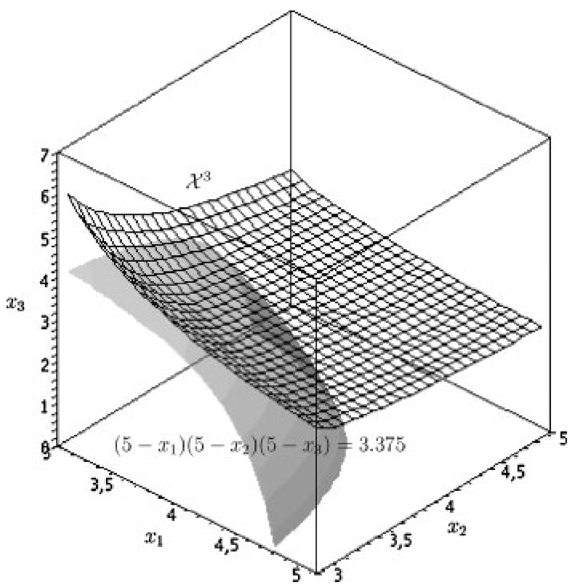

Figure 3. Region $\mathcal{X}^{(3)}$ and the optimal level set.

4.2. A solution concept for the general case

We propose the Lagrange multiplier method to solve optimisation problem (P) for $\beta = 0$. For the general case the NBS is characterised in the subsequent corollary. The longish proof is given in the Appendix. Numerical solutions can be easily obtained using Ref. [22].

Corollary 2. *Necessary and sufficient for* $\mathbf{x}^* = (x_1^*, \ldots, x_T^*)$ *to be the unique NBS is the following. There exist positive Lagrange multipliers* $\mu_1, \ldots, \mu_T$ *such that for each* $i \in \{1, \ldots, T\}$

(a) $\quad x_i^* \in \left\{ x_{i,\min}, -\frac{\alpha_i}{2} + \frac{1}{2}\sqrt{\alpha_i^2 + 4\alpha_i x_{i,\max}} \right\}$
with $\alpha_i = \sum_{j=1}^{T} \mu_j + \mu_i \xi_i$

(b) $\quad \left(1 - \sum_{j=1}^{T} \frac{1}{x_j^*} - \frac{\xi_i}{x_i^*}\right)\mu_i = 0.$

Corollary 2 provides a characterisation of data rates, that are optimal in the Nash bargaining sense. This is a global approach, where the receiver needs information from all users.

4.3. Decentralisation by a noncooperative game

We now deal with the issue of how we can define a local optimisation problem for each user. We utilise a technique which is well known in the theory of nonlinear programming as the concept of penalties, cf. Ref. [8]. As we use negative penalties in our context, we prefer to refer to them as incentives. In the local model, each user may optimise only its own parameters. Unrestricted data rates cannot be offered to each user. Giving incentives to the users to use a small data rate yields a Pareto-optimal point, as is shown in the following.

We introduce a set of positive parameters, denoted by θ_i, $1 \le i \le T$. User i with data rate r_i receives an incentive of $\theta_i\left[\omega/(\gamma_i r_i) + 1\right] = \theta_i x_i$. The intention of each user is to maximise its utility, which is defined as the sum of the utility and the incentive corresponding to data rate r_i. The utility of some user i corresponding to its data rate is defined as $\ln(x_{i,\max} - x_i)$. Taking the logarithm for the utility, but not for the incentive corresponding to some data rate, stresses the importance of the incentive and urges users to get by with low rates.

This leads to the following optimisation problem (U_i) for user i, $1 \le i \le T$

$$\begin{aligned} \text{maximize} \quad & \prod_{i=1}^{T} \ln\left(x_{i,\max} - x_i\right) + \theta_i x_i \\ \text{subject to} \quad & x_i - x_{i,\min} \ge 0 \\ & x_{i,\max} - x_i \ge 0. \end{aligned}$$

The network's aim is to give as little incentive to the users as possible. Therefore, the network's optimisation problem, called (N), is as follows.

$$\begin{aligned} \text{maximize} \quad & -\sum_{i=1}^{T} \theta_i x_i \\ \text{subject to} \quad & 1 - \sum_{j=1}^{T} \frac{1}{x_j} - \frac{\xi_i}{x_i} \ge 0, \ 1 \le i \le T \\ & x_i - x_{i,\min} \ge 0, \ 1 \le i \le T \\ & x_{i,\max} - x_i \ge 0, \ 1 \le i \le T \end{aligned}$$

If the incentive parameters $\theta_1, \ldots, \theta_T$ are properly chosen, the unique NBS of the global problem (P) solves the user problem (U_i) for each user $i \in \{1, \ldots, T\}$ and the network's problem (N). The proof of the following theorem is given in the Appendix.

Theorem 3 *Let $\boldsymbol{x}^*$ be the unique NBS of problem (P) and set*

$$\theta_i = \begin{cases} 2\left(\alpha_i - \sqrt{\alpha_i^2 + 4\alpha_i x_{i,\max}} + 2x_{i,\max}\right)^{-1} \\ \text{if } x_i^* = -\frac{\alpha_i}{2} + \frac{1}{2}\sqrt{\alpha_i^2 + 4\alpha_i x_{i,\max}} \\ \alpha_i / x_{i,\min}^2 + \tilde{\mu}_i \\ \text{if } x_i^* = x_{i,\min}, \end{cases}$$

with α_i and $\tilde{\mu}_i$ given in the proof of Corollary 2, $1 \leq i \leq T$. Then $\boldsymbol{x}^*$ solves the optimisation problem (N) and x_i^* solves (U_i) for each $1 \leq i \leq T$

The theorem states that the decentralised user problems yield a Pareto optimal point, the same optimal point as the global problem (P), if the incentive parameters a properly chosen.

5. A NONCOOPERATIVE INTERPRETATION

Another interesting game-theoretic interpretation is that the presented decentralisation results in a noncooperative game.

A noncooperative game consists of players, strategies and utility functions. Let $\mathcal{G} = \left(\mathcal{A}, \prod_{i\in\mathcal{A}} \mathcal{S}_i, \{u_i(.)\}_{i\in\mathcal{A}}\right)$, denote the noncooperative game where $\mathcal{A} = \{1, 2, \ldots, m\}$ is the set of players, $\mathcal{S}_i \subset \mathbb{R}^{m_i \times n_i}$, $m_i, n_i \in \mathcal{N}$, the strategy set of player i and $u_i : \prod_{i\in\mathcal{A}} \mathcal{S}_i \to \mathbb{R}$ the utility function of player i. $\prod_{i\in\mathcal{A}} \mathcal{S}_i$ denotes the Cartesian product of $\mathcal{S}_1, \ldots, \mathcal{S}_m$.

The solution concept that is most widely used for such games is the Nash equilibrium. A strategy $\tilde{\boldsymbol{x}} \in \prod_{i\in\mathcal{A}} \mathcal{S}_i$ is called Nash equilibrium if for every $i \in \mathcal{A}$

$$u_i(\tilde{x}_i, \tilde{\boldsymbol{x}}_{-i}) \geq u_i(x_i, \tilde{\boldsymbol{x}}_{-i}) \quad \text{for every } x_i \in \mathcal{S}_i$$

where $\tilde{\boldsymbol{x}}_{-i} = (\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \ldots, \tilde{x}_n)$. The interpretation is that, given the strategies of the other players, no user can improve its utility by making individual changes. This corresponds to a player optimising its performance regardless of the performance of other players. No one of the players has an incentive to deviate from a Nash equilibrium. However, a Nash equilibrium is not necessarily Pareto optimal.

In accordance with the above notation, the underlying game of the decentralisation is given by

$$\mathcal{G} = \left(\mathcal{A}, \prod_{i\in\mathcal{A}} \mathcal{S}_i, \{u_i(.)\}_{i\in\mathcal{A}}\right)$$

with

$$\begin{gathered} \mathcal{A} = \{1, \ldots, T\}, \\ \mathcal{S}_i = \left\{ \boldsymbol{x} \in \mathbb{R}^T \,\middle|\, x_{i,\min} \leq x_i \leq x_{i,\max} \right\}, \quad 1 \leq i \leq T, \\ u_i(\boldsymbol{x}) = \ln\left(x_{i,\max} - x_i\right) + \theta_i x_i, \quad 1 \leq i \leq T. \end{gathered}$$

The noncooperative game consists of T players. The strategy set of user i is the set of all vectors $\boldsymbol{x}$ such that the ith component lies between the minimum and maximum value of user i. The utility function is the sum of utility and incentive corresponding to some data rate.

Seeking for the Nash equilibrium, we are looking for some $\boldsymbol{x}^{\mathrm{Ne}} = \left(x_1^{\mathrm{Ne}}, \ldots, x_T^{\mathrm{Ne}}\right)$ with

$$\ln\left(x_{i,\max} - x_i^{\mathrm{Ne}}\right) + \theta_i x_i^{\mathrm{Ne}} \geq \ln(x_{i,\max} - x_i) + \theta_i x_i$$

for all $\boldsymbol{x} \in \prod_{j\in\mathcal{A}} \mathcal{S}_j$, $1 \leq i \leq T$. Thus, we need to find the maximum of the function $h(x) = \ln(x_{i,\max} - x) + \theta_i x$. The first derivative vanishes and the second derivative is positive for $x = x_{i,\max} - 1/\theta_i$. Taking into account the domain $\prod_{i\in\mathcal{A}} \mathcal{S}_i$ of $u_i(\boldsymbol{x})$, the unique Nash equilibrium of the game $\mathcal{G}$ is

$$\begin{gathered} \boldsymbol{x}^{\mathrm{Ne}} = \left(x_1^{\mathrm{Ne}}, \ldots, x_T^{\mathrm{Ne}}\right) \quad \text{with} \\ x_i^{\mathrm{Ne}} = \max\left\{x_{i,\min}, x_{i,\max} - \frac{1}{\theta_i}\right\}. \end{gathered}$$

The Nash equilibrium is Pareto optimal as the utility of each user is maximal at $\boldsymbol{x}^{\mathrm{Ne}}$. With the incentives chosen according to Theorem 3, the Nash equilibrium of the game $\mathcal{G}$ is the NBS $\boldsymbol{x}^{\mathrm{Ne}}$ to the problem (P). Thus, the decentralisation is a noncooperative implementation of a system's optimal point.

6. NUMERICAL RESULTS

To analyse the diverse advantages of the different solutions, we compare the sum of utility, the fairness and the total revenue of the above concepts. Figure 4 depicts the sum of utilities versus ξ_1 with $\xi_2 = 0.5$ fixed for a two user case. It can be seen that the NBS always lies between the solution to the sum-rate and the max–min problem. This holds because the max–min approach aims not at optimising some overall value but at avoiding single bad performance peaks. In that sense, this approach is fairer than the sum-rate approach. The performance loss of the NBS is small compared to the sum-rate approach, but it maintains the fairness in the presented game theoretic sense. Figure 5 presents the fairness of the different approaches, where fairness here means the ratio of x_1 and x_2 again for a two-user scenario where ξ_2

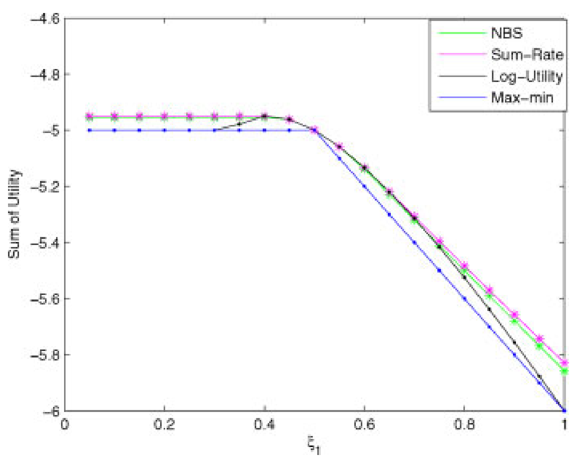

Figure 4. A two-person bargaining problem.

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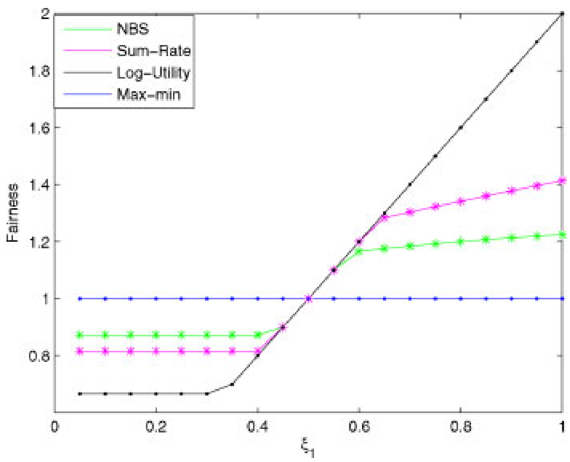

Figure 5. Fairness analysis of the different approaches.

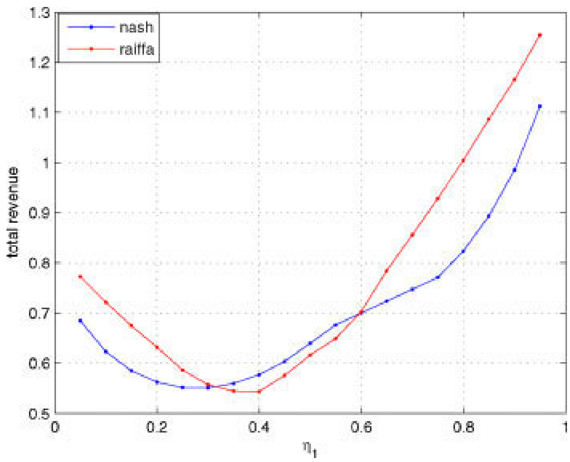

Figure 6. The total revenue of the Raiffa and NBS.

is set to 0.5. When ξ_1 is close to ξ_2 the performance of the approaches are the same. This fits to the above mentioned symmetry property. For the max–min problem, the ratio always equals one, which means both users have the small utility. The NBS lies between the sum-rate and the max–min problem. For the log-utility problem, the investigated ration shows the most variation for changing ξ_1. Finally, we compare the revenue of the different concepts. The total revenue of the game is defined as the sum of rates with respect to the willingness to pay

$$\sum_{i=1}^{T} r_i \hat{\eta}_i.$$

Figure 6 compares the total revenue of the Nash and the RBS for a two user case. It depends on the willingness to pay of each user which outcome gives a higher total revenue. For high or low willingness, Raiffa is better, whereas a user with medium willingness better goes for the NBS. Calculating further the total revenue of the remaining three approaches, we obtain Figure 7. Again, the outcome depends on the willingness to pay. Interestingly no concepts is outstanding

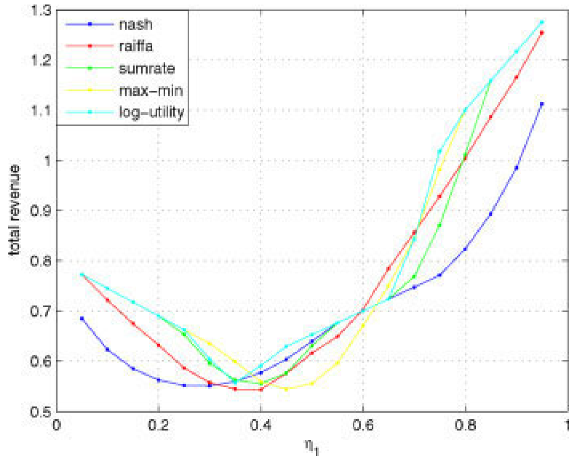

Figure 7. The total revenue of the five fairness concepts.

for all ranges. Those concepts that are better for a high or low willingness to pay perform worse in a medium range.

7. CONCLUSION

In this paper, we investigated fairness concepts for rate allocation in interference limited systems with rate adaption. Starting from general fairness concepts like the max–min fairness, we presented a cooperative game theoretical approach. In this game fairness is ensured by three axioms, while further efficiency is derived from PO. Both the NBS and the RBS have been investigated. To overcome the challenge of cooperation, a decentralisation has been proposed by exploiting so-called incentives. If the incentives a properly chosen, the decentralised solution coincides with the centralised one. Further, the decentralisation has an interesting interpretation as a noncooperative game, where the underlying Nash equilibrium was shown to be Pareto optimal. This proves that the system's optimal point can be decentrally determined by finding optimal strategies in a noncooperative game. Numerical results compare the fairness and efficiency of the proposed approaches. The game theoretic solutions turn out to be reliable candidates both for fairness and throughput, whereas the nongame theoretic solution have clear disadvantages. An open task which is dedicated to future research is the modelling of rate-adaption for multi-cell networks thereby taking into account the various interfering factors.

APPENDIX

Proof of corollary 2

Our aim is to use the Kuhn-Tucker Theorem, see, e.g. Ref. [23]. If $x_i = x_{i,\max}$ for some i the objective function in (P) is equal to zero. Thus, the solution satisfies $x_i^* < x_{i,\max}$ for all $1 \le i \le T$. The ln-function is strictly monotonic increasing,

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so we can consider the objective function

$$\ln\left(\prod_{i=1}^{T}(x_{i,\max}-x_i)\right)=\sum_{i=1}^{T}\ln(x_{i,\max}-x_i).$$

As the objective function is strictly concave and as the constraints in (P) are concave the Kuhn-Tucker conditions are necessary and sufficient for optimality, see Ref. [24].

Hence, we get the following necessary and sufficient conditions for the optimum $\boldsymbol{x}^*$. There exist Lagrange multipliers $\mu_i, \tilde{\mu}_i, \breve{\mu}_i \geq 0$, $1 \leq i \leq T$ such that

$$\nabla h(\boldsymbol{x}^*)=\sum_{i=1}^{T}\mu_i\nabla g_i(\boldsymbol{x}^*)+\sum_{i=1}^{T}\tilde{\mu}_i\nabla\tilde{g}_i(\boldsymbol{x}^*)+\sum_{i=1}^{T}\breve{\mu}_i\nabla_i(\boldsymbol{x}^*) \tag{5}$$

$$\begin{aligned} &g_i(\boldsymbol{x}^*)\mu_i=0, \quad \tilde{g}_i(\boldsymbol{x}^*)\tilde{\mu}_i=0\\ &_i(\boldsymbol{x}^*)\breve{\mu}_i=0, \quad 1\leq i\leq T \end{aligned} \tag{6}$$

where

$$\begin{aligned} &h(\boldsymbol{x})=-\sum_{i=1}^{T}\ln(x_{i,\max}-x_i),\\ &g_i(\boldsymbol{x})=1-\sum_{j=1}^{T}\tfrac{1}{x_j}-\tfrac{\xi_i}{x_i},\\ &\tilde{g}_i(\boldsymbol{x})=x_i-x_{i,\min}, \quad _i(\boldsymbol{x})=x_{i,\max}-x_i \end{aligned} \tag{7}$$

for $1 \leq i \leq T$. Again, noting that $x_i < x_{i,\max}$ we get $\{i \in \{1,\ldots,T\}\,|_i(\boldsymbol{x}^*)=0\}=\emptyset$ and $\breve{\mu}_i=0$ for all $1\leq i\leq T$. Differentiating $h(\boldsymbol{x})$, $g_i(\boldsymbol{x})$ and $\tilde{g}_i(\boldsymbol{x})$, Equation (5) transforms to

$$\frac{1}{x_{i,\max}-x_i^*}=\frac{1}{x_i^{*2}}\left(\sum_{j=1}^{T}\mu_j+\mu_i\xi_i\right)+\tilde{\mu}_i, \quad 1\leq i\leq T. \tag{8}$$

Let $\alpha_i=\left(\sum_{j=1}^{T}\mu_j+\mu_i\xi_i\right)$, then Equation (8) yields

$$\tilde{\mu}_i x_i^{*3}+\left(1-\tilde{\mu}_i x_{i,\max}\right)x_i^{*2}+\alpha_i x_i^*-\alpha_i x_{i,\max}=0, \quad 1\leq i\leq T. \tag{9}$$

Further, using Equation (7), Equation (6) reads as

$$\left(1-\sum_{j=1}^{T}\frac{1}{x_j^*}-\frac{\xi_i}{x_i^*}\right)\mu_i=0, \quad \left(x_i^*-x_{i,\min}\right)\tilde{\mu}_i=0.$$

Thus, either $x_i^*=x_{i,\min}$ or $\tilde{\mu}_i=0$. If $\tilde{\mu}_i=0$, (9) yields $x_i^{*2}+\alpha_i x_i^*-\alpha_i x_{i,\max}=0$, which transforms to $x_i^*=-\frac{\alpha_i}{2}\pm\frac{1}{2}\sqrt{\alpha_i^2+4\alpha_i x_{i,\max}}$ and the assertion follows as $x_i^*\geq x_{i,\min}>0$.

Proof of theorem 3

First, we prove that the NBS $\boldsymbol{x}^*$ solves (N). We exploit the Kuhn-Tucker Theorem as shown in the proof of Corollary 2 and define

$$\begin{aligned} &h(\boldsymbol{x})=\sum_{i=1}^{T}\theta_i x_i, \quad g_i(\boldsymbol{x})=1-\sum_{j=1}^{T}\tfrac{1}{x_j}-\tfrac{\xi_i}{x_i}\\ &\tilde{g}_i(\boldsymbol{x})=x_i-x_{i,\min}, \quad _i(\boldsymbol{x})=x_{i,\max}-x_i \end{aligned}$$

for all v$g_i \leq i \leq T$. Let η_i, $\tilde{\eta}_i$ and $\breve{\eta}_i$ be the corresponding Lagrange multipliers. Denote by $\mathbf{x}^{\text{net}}$ a solution to the network problem (N). Substituting the derivative of h, g_i, $\tilde{g}_i$ and $\breve{g}_i$, $1\leq i\leq T$, in Equation (5), it follows that

$$\theta_i=\left(\sum_{i=1}^{T}\eta_j+\xi_i\eta_i\right)\frac{1}{\left(x_i^{\text{net}}\right)^2}+\tilde{\eta}_i-\breve{\eta}_i, \quad 1\leq i\leq T.$$

As the solution must be positive it transpires that

$$x_i^{\text{net}}=\sqrt{\frac{\alpha_i'}{\theta_i-\tilde{\eta}_i+\breve{\eta}_i}} \quad \text{for all } 1\leq i\leq T$$

where $\alpha_i'=\sum_{j=1}^{T}\eta_j+\xi_i\eta_i$. Applying the Kuhn-Tucker Theorem, the following conditions are necessary and sufficient for an optimum of (N). For all $1\leq i\leq T$ it holds that

$$x_i^{\text{net}}=\sqrt{\frac{\alpha_{i'}}{\theta_i-\tilde{\eta}_i+\breve{\eta}_i}} \tag{10}$$

$$\left(1-\sum_{j=1}^{T}\frac{1}{x_j^{\text{net}}}-\frac{\xi_i}{x_i^{\text{net}}}\right)\eta_i=0 \tag{11}$$

$$\left(x_i^{\text{net}}-x_{i,\min}\right)\tilde{\eta}_i=0$$

$$\left(x_{i,\max}-x_i^{\text{net}}\right)\breve{\eta}_i=0$$

Now assume $\eta_i=\mu_i$, $\breve{\eta}_i=0$ and $\tilde{\eta}_i=\tilde{\mu}_i$ with μ_i and $\tilde{\mu}_i$ from the proof of Corollary 2. Then x_i^* fulfills Equation (11) for all $1\leq i\leq T$. It remains to show the validity of Equation (10). If $\tilde{\mu}_i=0$, we receive $\tilde{\eta}_i-\breve{\eta}_i=0$. Furthermore, Corollary 2 tells us

$$x_i^*=-\frac{\alpha_i}{2}+\frac{1}{2}\sqrt{\alpha_i^2+4\alpha_i x_{i,\max}}\,.$$

We show that $x_i^* = -\frac{\alpha_i}{2} + \frac{1}{2}\sqrt{\alpha_i^2 + 4\alpha_i x_{i,\max}}$ fulfills Equation (10). With $\alpha_i' = \alpha_i$ Equation (10) transforms to

$$\begin{aligned} x_i^{\text{net}} &= \sqrt{\frac{\alpha_i'}{\theta_i}} \\ &= \sqrt{\frac{\alpha_i}{2}\left(\alpha_i - \sqrt{\alpha_i^2 + 4\alpha_i x_{i,\max}} + 2x_{i,\max}\right)} \\ &= \sqrt{\left(\frac{\alpha_i^2}{4} - \frac{\alpha_i}{2}\sqrt{\alpha_i^2 + 4\alpha_i x_{i,\max}} + \left(\frac{\alpha_i^2}{4} + \alpha_i x_{i,\max}\right)\right)} \\ &= \sqrt{\left(-\frac{\alpha_i}{2} + \frac{1}{2}\sqrt{\alpha_i^2 + 4\alpha_i x_{i,\max}}\right)^2} \\ &= -\frac{\alpha_i}{2} + \frac{1}{2}\sqrt{\alpha_i^2 + 4\alpha_i x_{i,\max}} = x_i^*. \end{aligned}$$

The case $\tilde{\mu}_i \neq 0$ for some $1 \leq i \leq T$ is obvious as it simplifies to $x_i^* = x_{i,\min}$. This proves that the NBS solves (N).

It remains to show that the NBS x_i^* solves (U_i), too. In what follows, we assume that $i \in \{1, \ldots, T\}$ is arbitrary but fixed. Calculating the roots of the derivative of

$$\tilde{h}(x_i) = \ln\left(x_{i,\max} - x_i\right) + \theta_i x_i$$

yields $x_i = x_{i,\max} - 1/\theta_i$. As $D^2\tilde{h}(x_i) = -(x_{i,\max} - x_i)^{-2} < 0$, we observe that $x_{i,\max} - 1/\theta_i$ solves the optimisation problem if it fulfills the constraints, i.e. if $x_{i,\min} \leq x_{i,\max} - 1/\theta_i \leq x_{i,\max}$. Noting that $\theta_i > 0$, this simplifies to $x_{i,\max} - 1/\theta_i \geq x_{i,\min}$. As $f(x_i)$ is strictly monotonic decreasing for all $x_i < x_{i,\max} - \frac{1}{\theta_i}$ and strictly monotonic increasing for all $x_i > x_{i,\max} - \frac{1}{\theta_i}$, the solution to the optimisation problem (U_i) is given by

$$x_i^{\text{user}} = \begin{cases} x_{i,\max} - \frac{1}{\theta_i}, & \text{if } x_{i,\min} < x_{i,\max} - \frac{1}{\theta_i} \\ x_{i,\min}, & \text{if } x_{i,\min} \geq x_{i,\max} - \frac{1}{\theta_i} \end{cases} \tag{12}$$

It remains to prove that $x_i^* = x_i^{\text{user}}$ for all $1 \leq i \leq T$. If $\tilde{\mu}_i \neq 0$, then $x_i^* = x_{i,\min}$. By Equation (12) we have to show that $x_{i,\min} \geq x_{i,\max} - (1/\theta_i)$. With $\theta_i = \alpha_i / x_{i,\min}^2 + \tilde{\mu}_i$ it holds that

$$\begin{aligned} & x_{i,\min} \geq x_{i,\max} - \frac{1}{\theta_i} \\ \Leftrightarrow\ & x_{i,\min} \geq x_{i,\max} - \frac{x_{i,\min}^2}{\alpha_i + \tilde{\mu}_i x_{i,\min}^2}. \end{aligned}$$

This transforms to

$$\tilde{\mu}_i x_{i,\min}^3 + \left(1 - \tilde{\mu}_i x_{i,\max}\right) x_{i,\min}^2 + \alpha_i x_{i,\min} - \alpha_i x_{i,\max} \geq 0$$

which is true by Equation (9).

If $\tilde{\mu}_i = 0$ then

$$x_i^* = -\frac{\alpha_i}{2} + \frac{1}{2}\sqrt{\alpha_i^2 + 4\alpha_i x_{i,\max}}.$$

It holds that

$$\begin{aligned} x_{i,\max} - \frac{1}{\theta_i} &= x_{i,\max} - \frac{1}{2}\left(\alpha_i - \sqrt{\alpha_i^2 + 4\alpha_i x_{i,\max}} + 2x_{i,\max}\right) \\ &= -\frac{\alpha_i}{2} + \sqrt{\alpha_i^2 + 4\alpha_i x_{i,\max}} = x_i^*. \end{aligned} \tag{13}$$

If $x_i^* = x_{i,\min}$, Equation (13) yields $x_{i,\max} - \frac{1}{\theta_i} \geq x_{i,\min}$ and with Equation (12) we obtain $x_i^{\text{user}} = x_{i,\min}$, so $x_i^* = x_i^{\text{user}}$. If $x_i^* > x_{i,\min}$, Equation (13) yields $x_{i,\max} - \frac{1}{\theta_i} > x_{i,\min}$, and again with Equation (12) we observe $x_i^{\text{user}} = x_{i,\max} - \frac{1}{\theta_i} = x_i^*$. This concludes the proof.

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