A Game Theoretic Approach to Capacity Sharing in CDMA Radio Networks

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Abstract—**A fundamental problem in code division multiple access (CDMA) systems is the study of the capacity region and its optimal utilization by a community of users. In this paper, we use cooperative game theory to analyze the capacity region. We introduce utility functions and formulate a Nash bargaining problem in order to find an optimal element. The Nash bargaining solution is uniquely defined by four axioms. In this way, efficiency and fairness is obtained. It is shown that this global approach can be decentralized while the solution remains the same, which is important for practical application.**

I. Introduction

The expensive investment in new UMTS networks requires sophisticated techniques to guarantee an efficient use of the scarce available resources. Unlike TD/FDMA systems the capacity of a base station in a CDMA system is interference limited.

Early analytical work on the capacity of a CDMA system is [2], where the probability to maintain the signal-to-interference requirements is computed. In [11], the authors extend the work by calculating the total interference numerically. In [5], the admissible transmission rates for a fixed number of users are described as a convex set. This forms the starting point for the present work. However, in [5] the results are restricted to just one base station. An extension is given in [1] where the capacity region for several cells is defined and an interesting duality between the up- and downlink solutions is revealed. In [4] the geometrical properties of the capacity region are analyzed and optimal power assignment schemes are investigated. Our approach is different from the one in [4]. We use cooperative game theory to determine an optimal power allocation. By using the theory developed in [10] we obtain a Nash bargaining solution which provides the desired properties. In this paper, bargaining theory is applied to a general convex optimization problem. We also use techniques related to [12] where a game-theoretic framework for bandwidth allocation in broadband networks is considered. Reference [9] was one of the first to apply game theoretic methods to the problem of power control. A noncooperative power control game is proposed where the outcome is a Nash equilibrium. In [3] a similar game is formulated in terms of known instead of complex network parameters. Further the approach is extended to the case of multiple base stations. One deficiency of these games is that the resulting Nash equilibrium is not Pareto optimal in general. In this paper, we propose a cooperative game to chose an element of the capacity region. In particular, by using bargaining theory, we obtain a Pareto optimal element which further fulfills axioms of efficiency and fairness and maximizes revenue. It is important to note that the Pareto optimal Nash bargaining solution and the Nash equilibrium are different, not related concepts.

This paper is organized as follows. In Section II we present the basic concepts of cooperative game theory. An application of bargaining theory to a general convex optimization problem is given in Section III. Section IV constitutes the paper's core. After introducing the capacity region for a UMTS-network, we use the game-theoretic framework to choose a suitable element of this region. Both a centralized and a decentralized approach are considered. We present an overview of the results and some proposals for possible extensions in Section V.

II. Basic Concepts form Game Theory

We start by reviewing some basic concepts of cooperative bargaining theory [7]. An n-person bargaining problem is a pair $(U, \boldsymbol{u}^0)$, where $U \subset \mathbb{R}^n$ is a nonempty convex closed and upper bounded set and $\boldsymbol{u}^0 = (u_1^0, \ldots, u_n^0) \in \mathbb{R}^n$ such that $\boldsymbol{u} \geq \boldsymbol{u}^0$ componentwise for some $\boldsymbol{u} = (u_1^0, \ldots, u_n^0) \in U$, see Fig. 1. The elements of U are called outcomes and $\boldsymbol{u}^0$ is the disagreement outcome. The interpretation of such a problem is as follows. A number of n bargainers are faced with the problem to negotiate for a fair point in the convex set U. If no agreement can be achieved by the bargainers, the disagreement utilities $u_1^0, \ldots, u_n^0$ will be the outcome of the game. Let B_n denote the family of all n-person bargaining problems.

A bargaining solution is a function $F : B_n \to \mathbb{R}^n$ such that $F(U, \boldsymbol{u}^0) \in U$ for all $(U, \boldsymbol{u}^0) \in B_n$. Nash suggested a solution that is based on certain axioms, as given below.

(WPO) *Weak Pareto optimality*: $F : B_n \to \mathbb{R}^n$ is called weakly Pareto optimal, if for all $(U, \boldsymbol{u}^0) \in B_n$ it holds that there exists no $\boldsymbol{u} \in U$ satisfying $\boldsymbol{u} > F(U, \boldsymbol{u}^0)$.

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(SYM) *Symmetry*: $F : B_n \to \mathbb{R}^n$ is symmetric if $F_i(U, \boldsymbol{u}^0) = F_j(U, \boldsymbol{u}^0)$ for all $(U, \boldsymbol{u}^0) \in B_n$ that are symmetric with respect to a subset $J \subseteq \{1, \ldots, n\}$ for all $i, j \in J$ (i.e., $u_i^0 = u_j^0$ and $(u_1, u_2, \ldots, u_{i-1}, u_j, u_{i+1}, \ldots, u_{j-1}, u_i, u_{j+1}, \ldots \ldots, u_n) \in U$ for all $i < j \in J, \boldsymbol{u} \in U$).

(SCI) *Scale covariance*: $F : B_n \to \mathbb{R}^n$ is scale covariant if $F(\varphi(U), \varphi(\boldsymbol{u}^0)) = \varphi(F(U, \boldsymbol{u}^0))$ for all $\varphi : \mathbb{R}^n \to \mathbb{R}^n, \varphi(\boldsymbol{u}) = \bar{\boldsymbol{u}}$ with $\bar{u}_i = a_i u_i + b_i$, $a_i, b_i \in \mathbb{R}, a_i > 0, i = 1, \ldots, n$.

(IIA) *Independence of irrelevant alternatives*: $F : B_n \to \mathbb{R}^n$ is independent of irrelevant alternatives, if $F(U, \boldsymbol{u}^0) = F(\bar{U}, \bar{\boldsymbol{u}}^0)$ for all $(U, \boldsymbol{u}^0), (\bar{U}, \bar{\boldsymbol{u}}^0) \in B_n$ with $\boldsymbol{u}^0 = \bar{\boldsymbol{u}}^0, U \subseteq \bar{U}_0$ and $F(\bar{U}, \bar{\boldsymbol{u}}^0) \in U$.

Remark 1: Weak Pareto optimality means that no bargainer can gain over the solution outcome. Symmetry, scale covariance and independence of irrelevant alternatives are the so called axioms of fairness. The symmetry property states that the solution does not depend on the specific label, i.e., users with both the same initial points and objectives will obtain the same performance. Scale covariance requires the solutions to be covariant under positive affine transformations. Independence of irrelevant alternatives demands that the solution outcome does not change when the set of possible outcomes shrinks but still contains the original solution.
These four axioms imply *Pareto optimality* which means that it is impossible to increase any player's utility without decreasing another player's utility.

The Nash bargaining solution is defined as follows.

Definition 2: A function $N : B_n \to \mathbb{R}^n$ is said to be a Nash bargaining solution (NBS) if

$$N(U, \boldsymbol{u}^0) = \operatorname{argmax}\Big\{ \prod_{\substack{1 \le j \le n \\ u_j \ne u_j^0}} (u_j - u_j^0) \,\big|\, \boldsymbol{u} \in U, \boldsymbol{u} \ge \boldsymbol{u}^0 \Big\},$$

if $U \backslash \{\boldsymbol{u}^0\} \ne \varnothing$ and $N(U, \boldsymbol{u}^0) = \boldsymbol{u}^0$, else.

The NBS calls for the maximization of the product of the users' gain from cooperation. In addition it is uniquely characterized by the four axioms stated above. The proof of the following theorem can be found in [10].

Theorem 3: Let $F : B_n \to \mathbb{R}^n$ be a bargaining solution. Then the following two statements are equivalent:

(a) $F = N$.
(b) F satisfies WPO, SYM, SCI, IIA.

III. The Bargaining Solution for a Multi-Objective Convex Programming Problem

The aim of this section is to use the game-theoretic framework explained above to calculate a fair and efficient solution for a multi-objective programming problem. In the next section we apply this setting to the problem of optimizing the admissible n-user region of a CDMA-system.

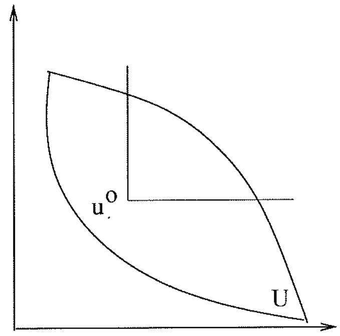

Fig. 1. A Nash bargaining problem

Let $X \subseteq \mathbb{R}^n$ be a nonempty closed convex set and $f_1, \ldots, f_k$ upper bounded real valued concave functions on X. It is often impossible to maximize these functions simultaneously on X. An appropriate alternative to find an eligible element in the set $f(X) = \{f(x) \mid x \in X\}$ is provided by the game theory. We formulate a bargaining problem and give a characterization for the NBS.

As the set $f(X)$ is not convex in general, we cannot choose it as the set of possible outcomes for our bargaining problem. Instead we consider the following set U, which contains $f(X)$, but is unbounded below and thus convex. Let

$$U = \{\boldsymbol{u} \in \mathbb{R}^n \mid \exists\, \boldsymbol{x} \in X \text{ s.t. } \boldsymbol{u} \le f(\boldsymbol{x})\}$$

where $f(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$. Further let $u^0 \in \mathbb{R}^n$ so that

$$X_0 = \{\boldsymbol{x} \in X \mid f(\boldsymbol{x}) \ge \boldsymbol{u}^0\} \ne \varnothing. \tag{1}$$

Proposition 4: $(U, \boldsymbol{u}^0)$ is a bargaining problem.

Proof: The set U is nonempty as X is nonempty. Furthermore, U is bounded as $f_1, \ldots, f_n$ are bounded. To show that U is convex let $\boldsymbol{u}, \tilde{\boldsymbol{u}} \in U$ and $\lambda \in (0, 1)$. Then there exists some $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in X$ such that $\boldsymbol{u} \le f(\boldsymbol{x})$, $\tilde{\boldsymbol{u}} \le f(\tilde{\boldsymbol{x}})$. Thus

$$\begin{aligned} \lambda \boldsymbol{u} + (1 - \lambda)\tilde{\boldsymbol{u}} &\le \lambda f(\boldsymbol{x}) + (1 - \lambda) f(\tilde{\boldsymbol{x}}) \\ &\overset{f \text{ concave}}{\le} f(\lambda \boldsymbol{x} + (1 - \lambda)\tilde{\boldsymbol{x}}). \end{aligned}$$

Further $\lambda \boldsymbol{x} + (1 - \lambda)\tilde{\boldsymbol{x}} \in X$ as X is convex. This yields $\lambda \boldsymbol{u} + (1 - \lambda)\tilde{\boldsymbol{u}} \in U$ which proves that U is convex.
From (1) it follows that $u \ge u^0$ for some $u \in U$ which concludes the proof. ■

Let $\boldsymbol{u}^* = N(U, \boldsymbol{u}^0)$ denote the NBS of the problem $(U, \boldsymbol{u}^0)$, cf. Definition 2. We call $X^* = \{\boldsymbol{x} \in X_0 \mid f(\boldsymbol{x}) = \boldsymbol{u}^*\}$ the Nash bargaining solution set, where X_0 is defined in (1). The set X^* consists exactly of those eligible elements of the set X we wanted to specify.

The subsequent theorem evinces how the Nash bargaining solution set can be calculated. The proof follows easily with Definition 2, cf. [10].

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Theorem 5: Let $J = \left\{ j \in \{1, \ldots, n\} \mid \max_{\boldsymbol{x} \in X_0} f_j(\boldsymbol{x}) > u_j^0 \right\}$. If $J = \varnothing$ then $\boldsymbol{u}^* = \boldsymbol{u}^0$ and $X^* = X_0$. If $J \neq \varnothing$ then $u_j^* = u_j^0$ for all $j \notin J$ and each element of the Nash bargaining solution set X^* satisfies the following problem

$$\max_{\boldsymbol{x} \in X_0} \prod_{j \in J} \left(f_j(\boldsymbol{x}) - u_j^0 \right).$$

IV. The NBS for the admissible n-user region

We consider the uplink of a CDMA system with chip rate ω, e.g., $\omega = 3.84$ MChip/s for UMTS. We confine ourselves to a single reference cell with an omnidirectional antenna. Assume there are n active users in the cell demanding for transmit capacity of $d_1, \ldots, d_n$ bits per second at an individual minimum bit error rate $e_1, \ldots, e_n$. Let $s_i = \omega / d_i$ denote the spreading gain. Since the bit error rate is a function of the bit energy-to-noise ratio, E_b/N_0, individual quality demands can be described by lower bounds e_i as follows.

$$\left(\frac{E_b}{N_0}\right)_i = s_i \left(\frac{C}{I}\right)_i \geq e_i, \quad i = 1, \ldots, n, \tag{2}$$

where C/I denotes the carrier-to-interference ratio at the base station.

Let p_i denote the transmit power of mobile i, and $a_i \in [0, 1]$ the transmission gain from mobile i to the base station.

We assume that $a_i > 0$ for all $i = 1, \ldots, n$, which obviously avoids meaningless assignments. Using the *effective spreading gain* $s_i' = s_i / e_i$, equation (2) reads as

$$s_i' \left(\frac{C}{I}\right)_i = s_i' \frac{p_i a_i}{\sum_{j \neq i} p_j a_j + \tau^0} \geq 1, \quad i = 1, \ldots, n. \tag{3}$$

The numerator $p_i a_i$ represents the received power of mobile i at the base station, $\sum_{j \neq i} p_j a_j$ collects the received interference from all other mobiles, and $\tau^0 > 0$ denotes the general background and thermal receiver noise at the base station. It also includes the system's pilot signal pollution.

The required quality-of-service e_i should be achieved at the minimum possible power level such that (3) is satisfied. Since the numerator of (3) is increasing in p_i and the denominator is increasing in p_j, $j \neq i$, it is clear that the minimum is attained at the boundary such that a solution $(p_1 \ldots, p_n)$ of the system

$$s_i' \frac{p_i a_i}{\sum_{j \neq i} p_j a_j + \tau^0} = 1, \quad i = 1, \ldots, n, \tag{4}$$

is needed. The unique solution is given as follows, for a proof see [5].

Theorem 6: The unique solution to (4) is given by

$$p_i = \frac{\tau_k}{a_i (s_i' + 1)\left(1 - \sum_{j \in C(k)} \frac{1}{s_j' + 1}\right)}, \quad i = 1, \ldots, n. \tag{5}$$

The analogue result has been shown for the case of multiple base stations [1]. Our restriction to one base station may be taken as an approximation for multiple base stations by subsuming all interference from other cells and the background noise into the single noise term τ.

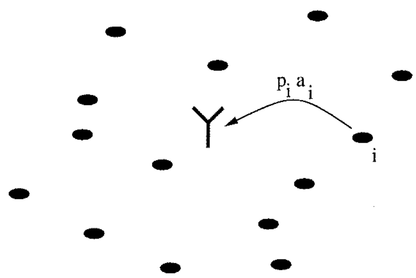

Fig. 2. One BS and several users

In what follows we assume that the transmit power for each mobile is bounded by p_{max}. Thus by (5) the set where n users are able to transmit at effective spreading gains s_i', $1 \leq i \leq n$, or corresponding data rates

$$d_i = \frac{\omega}{s_i' e_i}, \quad i = 1, \ldots, n,$$

is given by

$$A = \left\{ s' \middle| \frac{\tau}{a_i (s_i' + 1)\left(1 - \sum_{j=1}^{n} \frac{1}{s_j' + 1}\right)} \leq p_{max}, i = 1, \ldots, n \right\}$$

where $s' = (s_1', \ldots, s_n')$. The set A_n is called the admissible n-user region. Defining

$$x_i = s_i' + 1 = \frac{s_i}{e_i} + 1, \tag{6}$$

and $\gamma_i = \tau / (a_i p_{max})$ transforms A_n to

$$B = \left\{ \boldsymbol{x} \middle| x_i \left(1 - \sum_{j=1}^{n} \frac{1}{x_j}\right) \geq \gamma_i \ , i = 1, \ldots, n \right\},$$

with $\boldsymbol{x} = (x_1, \ldots, x_n)$.

We may assume that $d_i \in [d_{min}, d_{max}]$ with $d_{max} > d_{min} > 0$. Since $x_i = \omega / (d_i e_i) + 1$, it follows that

$$x_{i,min} \leq x_i \leq x_{i,max}$$

with $x_{i,min} = \omega / (d_{max} e_i) + 1$ and $x_{i,max} = \omega / (d_{min} e_i) + 1$. The set B is non-empty as $(\gamma_1 + n, \ldots, \gamma_n + n)$ lies in B. We further assume that d_{max} and d_{min} are chosen so that there exists an element in B which lies in $[x_{i,min}, x_{i,max}]$ componentwise.

In the following we analyze the constrained admissible n-user region

$$X = \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| 1 - \sum_{j=1}^{n} \frac{1}{x_j} - \frac{\gamma_i}{x_i} \geq 0, \right.$$
$$\left. x_{i,min} \leq x_i \leq x_{i,max} \text{ for all } 1 \leq i \leq n \right\},$$

with x_i defined in (6).

Theorem 7: X is convex, closed and nonempty.

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Proof: As $\frac{1}{x_i}$ is a convex function on $\mathbb{R}^n_+$ and as the sum of two convex functions is convex, we obtain the convexity of

$$-1+\sum_{j=1}^{n}\frac{1}{x_j}+\frac{\gamma_i}{x_i}, \quad 1\leq i\leq n,$$

on $\mathbb{R}^n_+$. The functions $-(x_i - x_{i,min})$, and $-(x_{i,max}-x_i)$ are convex as they are linear. The intersection of level sets of a convex function is convex, cf. [8, Corollary 4.6.1]. This yields the convexity of the sets

$$\Big\{\boldsymbol{x}\in\mathbb{R}^n_+ \mid 1-\sum_{j=1}^{n}\frac{1}{x_j}-\frac{\gamma_i}{x_i}\geq 0\Big\},$$
$$\{\boldsymbol{x}\in\mathbb{R}^n \mid x_i - x_{i,min}\geq 0\}, \text{ and}$$
$$\{\boldsymbol{x}\in\mathbb{R}^n \mid x_{i,max} - x_i\geq 0\}$$

for every $1\leq i\leq n$. Furthermore, the intersection of an arbitrary collection of convex sets is convex, cf. [8, Theorem 2.1]. Thus X is convex. Clearly X is closed and non-empty. ■

There are many solution concepts to single out a reasonable element of X, e.g. bargaining theory, proportional fairness, max-min fairness. Any one solution concept will usually violate the axioms associated with some other concept. We choose bargaining theory, because the properties of the resulting NBS are the most desirable in this model.

It is natural to assume that each user aims at obtaining a data rate greater than its minimum rate and as close to its maximum value as possible. If d_i tends to its maximum, then $-x_i$ tends to its maximum, too. Therefore, with respect to the framework described above, the performance function f_i for user i is defined as

$$f_i : X\to\mathbb{R}, \quad \boldsymbol{x}\mapsto -x_i. \tag{7}$$

Moreover

$$\boldsymbol{u}^0 = (-x_{1,max},\ldots,-x_{n,max}) \tag{8}$$

represents the initial or minimum performance. Referring to the game-theoretic framework in Section III we get

$$\begin{aligned} U &= \{\boldsymbol{u}\in\mathbb{R}^n \,|\, \exists\, \boldsymbol{x}\in X \text{ s.t. } f(\boldsymbol{x})\geq \boldsymbol{u}\},\\ X_0 &= \{\boldsymbol{x}\in X \,|\, f(\boldsymbol{x})\geq \boldsymbol{u}^0\} \end{aligned} \tag{9}$$

with $f = (f_i)_{i=1,\ldots,n}$. Using (7) and (6) we observe that $f(\boldsymbol{x})\geq \boldsymbol{u}^0$ transforms to $d_i\geq \omega/[(x_{i,max}-1)e_i]$, which means that for each user the minimum data rate $\omega/[(x_{i,max}-1)e_i] = d_{min}$ is ensured, as desired. This shows that the choice of u^0 in(8) sensible.
We are looking for an element of X which is a fair trade-off for each user. As explained above this leads to finding the NBS for the problem $(U,\boldsymbol{u}^0)$.

Lemma 8: The unique NBS to the bargaining problem $(U,\boldsymbol{u}^0)$ defined in (8) and (9) is the solution to the following convex optimization problem:

$$(P)\begin{cases}\max\prod_{i=1}^{n}(x_{i,max}-x_i)\\ \text{s.t.}\begin{cases}1-\sum_{j=1}^{n}\frac{1}{x_j}-\frac{\gamma_i}{x_i}\geq 0 & \text{for all } 1\leq i\leq n,\\ x_i - x_{i,min}\geq 0 & \text{for all } 1\leq i\leq n,\\ x_{i,max}-x_i\geq 0 & \text{for all } 1\leq i\leq n.\end{cases}\end{cases}$$

Proof: Since

$$J=\{j\in\{1,\ldots,n\} \mid \max_{x\in X_0} f_j(\boldsymbol{x}) > u^0_j\} = \{1,\ldots,n\}$$

the assertion follows from Theorem 5. ■

In general, a solution of (P) is hard to achieve. By using the Kuhn-Tucker theory the NBS may be characterized as follows.

Theorem 9: A necessary and sufficient condition for $\boldsymbol{x}^*=(x_1^*,\ldots,x_n^*)$ to be the unique NBS is the following. There exist $\mu_i\geq 0$, such that for each $1\leq i\leq n$

(a) $x_i^* \in \Big\{x_{i,min}, -\frac{\alpha_i}{2}\pm\frac{1}{2}\sqrt{\alpha_i^2+4\alpha_i x_{i,max}}\Big\}$ with $\alpha_i=\sum_{j=1}^{n}\mu_j+\mu_i\gamma_i$,

(b) $(1-\sum_{j=1}^{n}\frac{1}{x_j^*}-\frac{\gamma_i}{x_i^*})\mu_i = 0$,

(c) $|J_0(\boldsymbol{x}^*)|+|\tilde{J}_0(\boldsymbol{x}^*)| = n$,
where $J_0(\boldsymbol{x}^*)=\Big\{1\leq k\leq n \,\Big|\, 1-\sum_{j=1}^{n}\frac{1}{x_j^*}-\frac{\gamma_k}{x_k^*}=0\Big\}$
and $\tilde{J}_0(\boldsymbol{x}^*)=\{1\leq k\leq n \,|\, x_k^* = x_{k,min}\}$.

Proof: The main arguments of the proof are contributed by the Kuhn-Tucker theory. Therefore we have to show that the constraint qualification (CQ) holds, cf. [6]. The constraint functions in (P) are concave and the interior of X is non-empty. In this special case Slater has shown that (CQ) holds, cf. [6, Lemma 5.2].

If $x_i = x_{i,max}$ for some i the objective function in (P) equals zero. By assumption there exists an elements for which $\prod_{i=1}^{n}(x_{i,max}-x_i) > 0$. Thus, the solution to (P) satisfies $x_i\neq x_{i,max}$ and the second constraint yields

$$x_i > x_{i,max} \text{ for every } i=1,\ldots,n. \tag{10}$$

The function $f(\boldsymbol{x}) = \ln \boldsymbol{x}$ is strictly monotonic increasing, so we may instead consider the objective function

$$\ln\Big(\prod_{i=1}^{n}(x_{i,max}-x_i)\Big) = \sum_{i=1}^{n}\ln(x_{i,max}-x_i)$$

which is well defined by (10). By the Kuhn-Tucker Theorem we get the following necessary and sufficient conditions for the optimum $\boldsymbol{x}^*$. There exist $\mu_i,\tilde{\mu}_i,\bar{\mu}_i\geq 0$, $1\leq i\leq n$, such that

$$\begin{aligned} Df(\boldsymbol{x}^*) = &\sum_{i=1}^{n}\mu_i Dg_i(\boldsymbol{x}^*)+\sum_{i=1}^{n}\tilde{\mu}_i D\tilde{g}_i(\boldsymbol{x}^*)\\ &+\sum_{i=1}^{n}\bar{\mu}_i D\bar{g}_i(\boldsymbol{x}^*), \end{aligned} \tag{11}$$

$$g_i(\boldsymbol{x}^*)\mu_i = 0,\ \tilde{g}_i(\boldsymbol{x}^*)\tilde{\mu}_i = 0,\ \bar{g}_i(\boldsymbol{x}^*)\bar{\mu}_i = 0 \tag{12}$$

$$|J_0(\boldsymbol{x}^*)| + \big|\tilde{J}_0(\boldsymbol{x}^*)\big| + \big|\bar{J}_0(\boldsymbol{x}^*)\big| = n, \tag{13}$$

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where

$$f(\boldsymbol{x}) = -\sum_{i=1}^{n} \ln\left(x_{i,max} - x_i\right),$$

$$g_i(\boldsymbol{x}) = 1 - \sum_{j=1}^{n} \frac{1}{x_j} - \frac{\gamma_i}{x_i},\ \tilde{g}_i(\boldsymbol{x}) = x_i - x_{i,min},$$

$$\bar{g}_i(\boldsymbol{x}) = x_{i,max} - x_i \quad \forall 1 \le i \le n \qquad (14)$$

and $J_0(\boldsymbol{x}^*) = \{j \in \{1, \ldots, n\}\,|g_j(\boldsymbol{x}^*) = 0\}, \tilde{J}_0(\boldsymbol{x}^*)$ and $\bar{J}_0(\boldsymbol{x}^*)$ in an analogous way.

It remains to show that conditions (11)–(13) yield (a)–(c). Again, noting that $x_i < x_{i,max}$ we get $\bar{J}_0(\boldsymbol{x}^*) = \varnothing$ and $\bar{\mu}_i = 0$ for all $1 \le i \le n$. Therefore (c) follows with (13). Further, using (14) condition (12) simplifies to

$$\Big(1 - \sum_{j=1}^{n} \frac{1}{x_j^*} - \frac{\gamma_i}{x_i^*}\Big)\mu_i = 0, \qquad (15)$$

$$(x_i^* - x_{i,min})\,\tilde{\mu}_i = 0. \qquad (16)$$

Equation (15) gives (b). Condition (16) is equivalent to

$$x_i^* = x_{i,min} \text{ or } \tilde{\mu}_i = 0. \qquad (17)$$

Assume $\tilde{\mu}_i = 0$. We obtain x_i^* with (11). Differentiating $f(\boldsymbol{x}), g_i(\boldsymbol{x})$ and $\tilde{g}_i(\boldsymbol{x})$, equation (11) transforms to

$$\frac{1}{x_i^* - x_{i,max}} = \frac{1}{{x_i^*}^2}\Big(\sum_{j=1}^{n} \mu_j + \mu_i\gamma_i\Big), \quad 1 \le i \le n. \qquad (18)$$

Let

$$\alpha_i = \Big(\sum_{j=1}^{n} \mu_j + \mu_i\gamma_i\Big), \qquad (19)$$

then (18) yields

$$\left(1 - \tilde{\mu}_i x_{i,max}\right) {x_i^*}^2 + \alpha_i x_i^* + \alpha_i x_{i,max} = 0 \qquad (20)$$

for all $1 \le i \le n$. Thus,

$$x_i^* = -\frac{\alpha_i}{2} \pm \frac{1}{2}\sqrt{\alpha_i^2 + 4\alpha_i x_{i,max}} \text{ for all } 1 \le i \le n.$$

which gives together with (17) condition (a). This concludes the proof. ■

Theorem 9 provides a characterization of data rates, that are optimal in the Nash bargaining sense. The interpretation of Lagrange multiplier μ_i is the prize of data rate of user i.

The above approach requires complex communication between mobiles and base station. In the sequel we deal with an approach purely based on local problems for each mobile. We use a technique which is well known in the theory of nonlinear programming as the concept of penalties, cf. [12]. As we use negative penalties in our context, we rather refer to them as incentives. In the local model, each mobile may optimize only its own parameters. Unrestricted data rates cannot be offered to each mobile. Giving incentives to the mobiles to use a low data rate, yields a Pareto-optimal point, as is shown in the following.

We introduce n positive parameters, denoted by λ_i, $1 \le i \le n$. User i with data rate d_i receives an incentive of $\lambda_i\left[\omega/(d_i e_i) + 1\right]$. The intention of each mobile station is to maximize its utility which is defined as the sum of the utility and the incentive corresponding to data rate d_i.

The following optimization problem arises for each user i, $1 \le i \le n$.

$$(U_i)\begin{cases} \max \ln\left(x_{i,max} - x_i\right) + \lambda_i x_i \\ \text{s.t.} \begin{cases} x_i - x_{i,min} \ge 0, \\ x_{i,max} - x_i > 0. \end{cases} \end{cases}$$

The network's aim is to give as little incentives to the users as possible. Therefore the network's optimization problem is as follows.

$$(N)\begin{cases} \max -\sum_{i=1}^{n} \lambda_i x_i \\ \text{s.t.} \begin{cases} 1 - \sum_{j=1}^{n} \frac{1}{x_j} - \frac{\gamma_i}{x_i} \ge 0 & \text{for all } 1 \le i \le n, \\ x_i - x_{i,min} \ge 0 & \text{for all } 1 \le i \le n, \\ x_{i,max} - x_i \ge 0 & \text{for all } 1 \le i \le n. \end{cases} \end{cases}$$

The following theorem states that for certain incentive parameters the unique NBS of the global problem (P) solves the user problem (U_i) and the network's problem (N). The highly technical proof is omitted here.

Theorem 10: Let $\boldsymbol{x}^*$ be the unique NBS to the problem (P) and set

$$\lambda_i = \begin{cases} 2\left(\alpha_i - \sqrt{\alpha_i^2 + 4\alpha_i x_{i,max}} + 2x_{i,max}\right)^{-1}, \\ \qquad \text{if } x_i^* = -\frac{\alpha_i}{2} + \frac{1}{2}\sqrt{\alpha_i^2 + 4\alpha_i x_{i,max}}, \\ 2\left(\alpha_i + \sqrt{\alpha_i^2 + 4\alpha_i x_{i,max}} + 2x_{i,max}\right)^{-1}, \\ \qquad \text{if } x_i^* = -\frac{\alpha_i}{2} - \frac{1}{2}\sqrt{\alpha_i^2 + 4\alpha_i x_{i,max}}, \\ \alpha_i / x_{i,min}^2 + \beta_i, \text{ if } x_i^* = x_{i,min}, \end{cases}$$

with α_i given in Theorem 9, $1 \le i \le n$. Then $\boldsymbol{x}^*$ solves the optimization problem (N) and x_i^* solves (U_i) for every $1 \le i \le n$.

Implementation of the decentralized optimization problem requires knowledge of the variables α_i which are dependent on the Lagrange multipliers μ_i, cf. (19). These can be obtained using the gradient projection method. Having obtained the optimal element of the capacity region, the power allocation is calculated by (5). The comparison of diverse algorithms converging to the unique Nash bargaining vector will be the topic of further research.

The following remark shows that the decentralization is a noncooperative implementation of a point that is optimal in the Nash bargaining sense.

Remark 11: The presented decentralization results in a noncooperative game. A noncooperative game is a triple $(\mathcal{A}, \prod_{i\in\mathcal{A}} \mathcal{S}_i, \{u_i(.)\}_{i\in\mathcal{A}})$, where $\mathcal{A}$ is the set of players, $\mathcal{S}_i$ the strategy set of player i and $u_i : \prod_{i\in\mathcal{A}} \mathcal{S}_i \to \mathbb{R}$ the utility

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function of player i.
The underlying game of the above approach is

$$\Gamma = (\mathcal{A}, \prod_{i\in\mathcal{A}} \mathcal{S}_i, \{u_i(.)\}_{i\in\mathcal{A}})$$

with

$$\mathcal{A} = \{1, \dots, n\},$$
$$\mathcal{S}_i = \{\boldsymbol{x} \in \mathbb{R} \mid x_{i,\min} \le x_i \le x_{i,\max}\}, \quad 1 \le i \le n,$$
$$u_i(x) = \ln(x_{i,max} - x_i) + \lambda_i x_i, \quad 1 \le i \le n.$$

As mentioned in Section I, the solution that is most widely used for noncooperative games is the Nash equilibrium. A strategy $\boldsymbol{x}'$ is called Nash equilibrium if for every $i \in \mathcal{A}$

$$u_i(x'_i, \boldsymbol{x}'_{-i}) \ge u_i(x_i, \boldsymbol{x}'_{-i}) \quad \text{for every } u_i \in \mathcal{S}_i$$

where $\boldsymbol{x}'_{-i} = (x'_1, \dots, x'_{i-1}, x'_{i+1} \dots, x'_n)$. The interpretation is that given the strategies of the other players, no user can improve its utility by making individual changes. It is easy to see that our game Γ has a unique Nash equilibrium which is Pareto optimal. With the incentives chosen according to Theorem 10, the NBS x^* to the problem (P) is the Nash equilibrium of the game Γ. Thus, the decentralization is a noncooperative implementation of a system's optimal point.

V. Conclusion

In this paper, we propose an application of cooperative game theory to the analysis of the admissible n-user region. We show that the game-theoretic framework provides a promising and seminal approach to the question of fairness and efficiency. Fairness is ensured by three axioms, while efficiency is derived from a fourth condition on bargaining solutions.

The search for a data rate vector which is fair to all users is formulated as a bargaining problem. The unique NBS is characterized. As this global approach requires communication simultaneously between all mobile stations and the system, we decentralize the problem. It is written as an optimization problem for each user and, furthermore, as a network optimization problem employing so called incentives. The NBS solves these problems, if the incentive parameters are properly chosen so that candidates for the global solution may only be found in the local solution set.

Future research will deal with other solution concepts like the Kalai-Smorodinski solution, proportional fairness and max-min fairness. Furthermore, the investigations will be extended to the multi-cell framework.

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