Deterministic Bipolar Measurement Matrices with Flexible Sizes from Legendre Sequence

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Abstract

A deterministic method to construct bipolar measurement matrices for compressed sensing is proposed based on Legendre sequences. The novel matrices have remarkably flexible measurement sizes, relatively low coherence and show empirically good performance compared with Gaussian matrices.

Index Terms

Compressed Sensing, Legendre Sequences, Deterministic Method.

I. Introduction

By a simple "shift-and-add" of the Legendre sequence with a prime period P, $P = 3\ (mod\ 4)$, a family of binary signals with a low *maximum correlation magnitude* (for definition, see [1]) was first proposed by the two authors of this Letter [2]. Compared with the existing signal families with overwhelming periods of the form $2^r - 1$ [1], the family from Legendre sequence has a remarkably flexible periods, which significantly decreases the interval of two adjacent periods. A few years later, this family was found also naturally applicable for the case $P = 1\ (mod\ 4)$ [3], and then generalised to nonbinary signals [4]. Furthermore, the binary family was even practically used in the L1C signal design for GPS [5], owing to its flexible periods and favorable correlation properties.

In the past decade, compressed sensing (CS) has become a fascinating area, which deals with the reliable reconstruction of a k-sparse signal with length N from M measurements ($M < N$). How to construct suitable measurement matrices is one of the two main concerns in CS [6]. Based on the resultant family [2] from Legendre sequence, a novel method is proposed in this Letter to construct bipolar (± 1) measurement matrices with restricted isometry property (RIP) [6][7] for compressed sensing. One of the most remarkable features of the new construction is that, the measurements size (rows of matrices) can be *any* prime integer or *probably* an arbitrary even integer (no counterexamples for rows up to 10^4), which is not the case for most existing deterministic methods.

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II. Preliminaries

Legendre sequence $\mathbf{L}$ with a prime period P is defined by

Definition 1 [2]: $L(0) = 0$; for $1 \leq t \leq P-1$, $L(t) = 0$ if $t = x^2\ (mod\ P)$ for some $x \in \{1, 2, ..., P-1\}$, and $L(t) = 1$ otherwise. For $t \geq P$, define $L(t) = L(t\ (mod\ P))$.

Based on Legendre sequence with $P = 3\ (mod\ 4)$ and its decimation version, two signal sets are constructed (set $\mathbf{N}$ and set $\mathbf{M}$) by the "shift-and-add" operation in [2]. In this Letter, we only choose set $\mathbf{N}$ to construct measurement matrix, as set $\mathbf{N}$ includes more sequences than set $\mathbf{M}$. Besides, the decimation value (*dec* in [2]) is set to 1 here for simplicity. Therefore, set $\mathbf{N}$ can be defined by

Definition 2 [2]: Let $\mathbf{N} = \mathbf{L} \cup \mathbf{N}'$, where $\mathbf{N}' = \{L(t+i) \oplus L(t), t = 0, 1, ..., P-1\}$, $i = (P+1)/2, ..., P-1$.

Note that in Definition 2, '$\oplus$' denotes the addition modulo 2 and P can be any prime integer.

Remark 1: In [3], a set $\mathbf{X}' = \mathbf{L} \cup \mathbf{X}$ is defined. By setting $t' = t + i$, it is readily seen (from equ.8 [3]) that each signal with parameter i in $\mathbf{N}'$ is a shifted version of the one with parameter $P - i$ in $\mathbf{X}$, if the difference of $L(0)$ (for a binary signal, $L(0) = 1$ in [3]) is ignored. Due to the difference, there are two positions where the bits of a signal in $\mathbf{N}'$ and its counterpart in $\mathbf{X}$ are inverse. Nevertheless, since the proof of maximum correlation magnitude (denoted by R_{max}) for $\mathbf{X}'$ does not depend on the value of $L(0)$ [3], we have

Lemma 1: R_{max} of $\mathbf{N}$ is smaller than $2\sqrt{P} + 5$.

III. Construction

Suppose $\mathbf{S}$ is a set including Q binary signals with period P', $\{\mathbf{s}_0, \mathbf{s}_1, \cdots, \mathbf{s}_{Q-1}\}$, where $\mathbf{s}_j = [\mathbf{s}_j(0), \mathbf{s}_j(1), \cdots, \mathbf{s}_j(P'-1)]$, $0 \leq j \leq Q-1$. Define a mapping $f(y) = 1 - 2y$, $y \in \{0, 1\}$. Then a bipolar matrix $\mathbf{B}$ can be defined by $\mathbf{B} = (\mathbf{B}_0 | \mathbf{B}_1 | \cdots | \mathbf{B}_{Q-1})$, where

$$\mathbf{B}_j = \begin{pmatrix} f(\mathbf{s}_j(0)) & f(\mathbf{s}_j(P'-1)) & \cdots & f(\mathbf{s}_j(1)) \\ f(\mathbf{s}_j(1)) & f(\mathbf{s}_j(0)) & \cdots & f(\mathbf{s}_j(2)) \\ \vdots & \vdots & \ddots & \vdots \\ f(\mathbf{s}_j(P'-1)) & f(\mathbf{s}_j(P'-2)) & \cdots & f(\mathbf{s}_j(0)) \end{pmatrix} \tag{1}$$

From the definition of coherence (see [6], denoted by μ), it is obvious that μ of $\mathbf{B}$ is R_{max}/P'.

Lemma 2 [7][8]: A matrix with coherence μ satisfies RIP of order k with $\delta \leq \mu(k-1)$ whenever $k < 1/\mu + 1$. Moreover, every k-sparse signal can be exactly recovered by orthogonal matching pursuit (OMP), provided $k < \frac{1}{2}(1 + \frac{1}{u})$.

As a large k is desirable for compressed sensing, the set $\mathbf{S}$ with small R_{max} is preferred due to Lemma 2. Although some sets have an optimal R_{max} in terms of Welch bound [1], their periods are limited to $2^r - 1$, which put too severe a restriction on the rows of $\mathbf{B}$. In the following, we construct $\mathbf{B}$ with flexible rows from set $\mathbf{N}$ in Definition 2.

L-method I: Let $\mathbf{S} = \mathbf{N}$. For M prime, an $M \times N$ measurement matrix $\mathbf{A}$ can be directly obtained from Equ. (1) by selecting any N columns of $\mathbf{B}$, where $N \leq M(M+1)/2$. The first N columns are selected in this Letter.

L-method II: If M is a sum of two primes, P_1 and P_2, an $M \times N$ measurement matrix $\mathbf{A}$ ($N \le P_{min}(P_{min}+1)/2$, $P_{min} = min\{P_1, P_2\}$) can be obtained as follows. First, construct a $P_1 \times N$ matrix and a $P_2 \times N$ matrix from L-method I. Then place one matrix on the top of the other.

From Lemma 1 and the constructions above, it is clear that

Lemma 3: μ of $\mathbf{A}$ from L-method I is upper bounded by $(2\sqrt{M}+5)/M$; μ of $\mathbf{A}$ from L-method II is upper bounded by $2(\sqrt{P_1}+\sqrt{P_2}+5)/M$.

Generally, for a given even M, there are many ways to split M (see Fig.1, Left). To lower the upper bound of μ, $|P_2 - P_1|$ should be maximised subject to $P_{min}(P_{min}+1)/2 \ge N$. On the other hand, when $|P_2 - P_1| \le 2(\sqrt{P_1}+\sqrt{P_2}+5)$, the columns can be doubled by the following self-evident Lemma without increasing the upper bound of μ.

Lemma 4: If $\mathbf{A}_i$ is a bipolar $M_i \times N$ matrix with μ_i for $i = 1, 2$, then

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & -\mathbf{A}_1 \\ \mathbf{A}_2 & \mathbf{A}_2 \end{pmatrix} \tag{2}$$

is an $(M_1 + M_2) \times 2N$ matrix with $\mu \le \frac{1}{M_1+M_2} max\{\mu_1 M_1 + \mu_2 M_2, |M_1 - M_2|\}$.

IV. SIMULATIONS

In this section, the empirical performance of the new measurement matrices is illustrated by several examples. Simulation results below show that these matrices outperform the corresponding Gaussian random matrices under the OMP algorithm. The k-sparse signals are generated by firstly selecting the nonzero positions uniformly at random and then selecting values subject to the standard normal distribution $N(0, 1)$. For each measurement matrix and each k-sparse signal $\mathbf{x}$, 1000 Monte Carlo trials are run to obtain a smooth curve. In each trial, a recovery error $e = \|\mathbf{x}^* - \mathbf{x}\|_2$ is computed, where $\mathbf{x}^*$ denotes the recovered signal. If $e < 10^{-6}$, we declare this recovery to be perfect. Finally, a percentage of perfect recovery is obtained.

Example 1: Let $M = 263$ and $N = 600$. As M is prime and $N \le M(M+1)/2$, we use L-method I. A 263×600 matrix $\mathbf{A}$ is generated from the first 600 columns of the 263×34716 matrix $\mathbf{B}$. In Fig.2, we observe that the performance of the matrix from Legendre sequence exceeds that of the corresponding Gaussian matrix (denoted by 'Rand').

Example 2: Let $M = 500$ and $N = 1000$. As M is an even integer and M can be expressed as the sum of two prime integers, say $M = 229 + 271$, such that $N \le 229(229+1)/2 = 26335$, L-method II can be employed. Similarly, as $N \le 61(61+1)/2 = 1891$, the split $M = 61 + 439$ can also be used to construct the 500×1000 matrix. We see in Fig.3 that the two deterministic matrices both outperform their Gaussian counterpart. As expected, the deterministic matrix from the split $M = 61 + 439$ performs better than the one from the split $M = 229 + 271$, which can be partly explained by the smaller upper bound of μ for the former split.

Example 3: Let $M = 300$ and $N = 3000$. Since $M = 89 + 211$ and $N \le 89(89+1)/2 = 4005$, L-method II can be used to obtain $\mathbf{A}$. In Fig.4, we see that with a relatively larger ratio of $N/M = 10$, the matrix from Legendre sequence still outperforms the Gaussian matrix.

Example 4: To evaluated the performance loss of the L-method I/II, which is based on set **N** with a relatively low but not optimal R_{max}, we depict in Fig.5 the performance of matrices from Kasami set (small) [1], which is well known for its optimal R_{max}. we see that, as expected, the matrix from Kasami set outperforms the counterpart from set **N**. Nevertheless, it should be noted that M for Kasami set can only be of the form $2^{r'} - 1$ (r' even) and the maximal ratio of N/M is at most $\sqrt{M+1}$, which poses strict limitations on M and N. By contrast, the maximal ratio of N/M for L-method I is $(M+1)/2$. For L-method II, this ratio is depicted in Fig.1 (Right) in terms of $log_2(N/M)$, which is much greater than that of the matrix from Kasami set for $M \geq 255$.

V. Conclusion

Based on Legendre sequences, a deterministic construction for bipolar measurement matrices satisfying the RIP condition is presented for compressed sensing. The rows of these matrices can be *any* prime integer or *probably* an arbitrary even integer. Simulation results show that the novel bipolar measurement matrices have good performance compared with Gaussian matrices.

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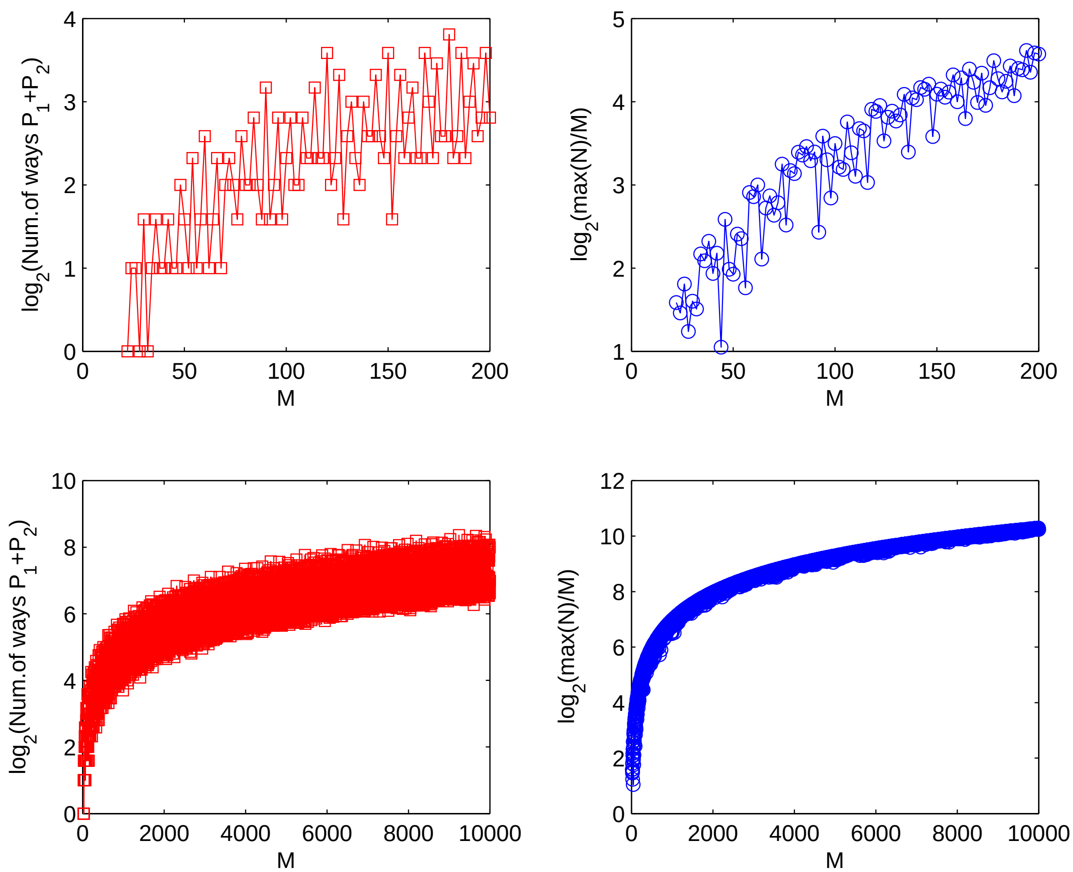

Fig. 1. Left: Num. of ways to express an even M as $P_1 + P_2$ ($22 \le M \le 10^4$, P_1 and P_2 are both prime integers such that $7 \le P_1 \le P_2$); Right: Maximal N available for M.

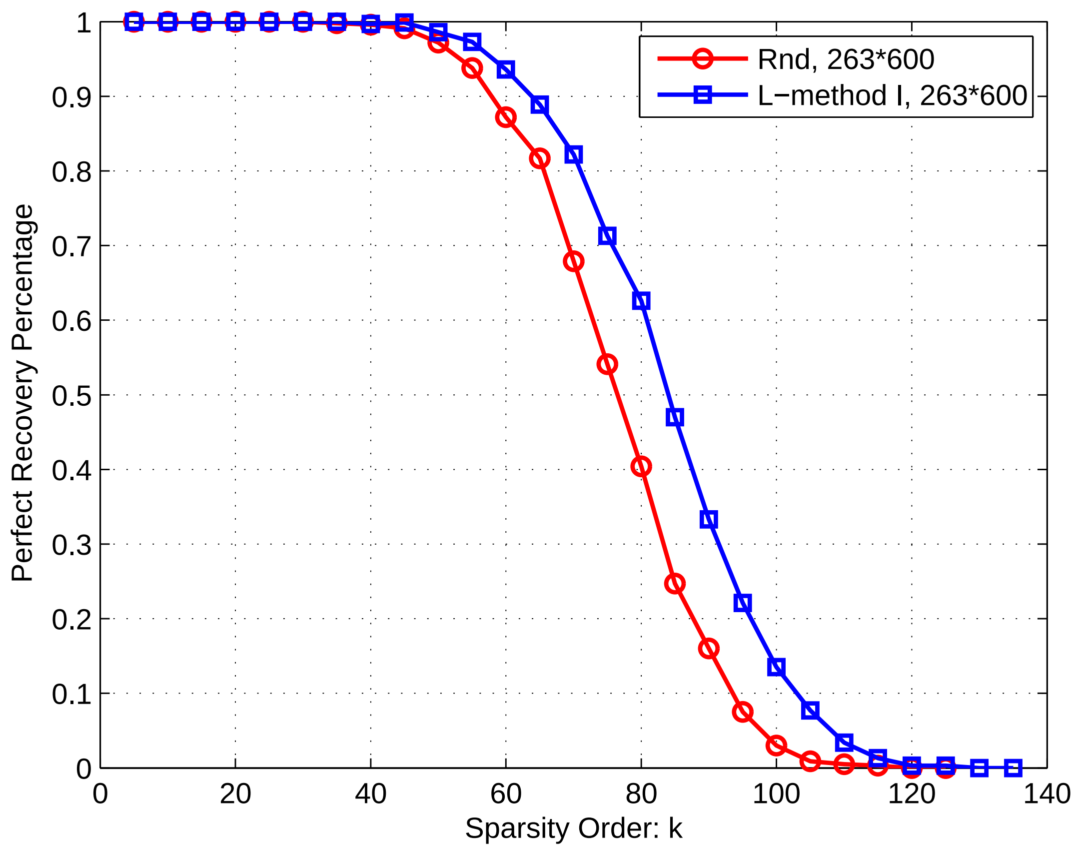

Fig. 2. Empirical performance of a measurement matrix from L-method I and the corresponding Gaussian matrix under OMP

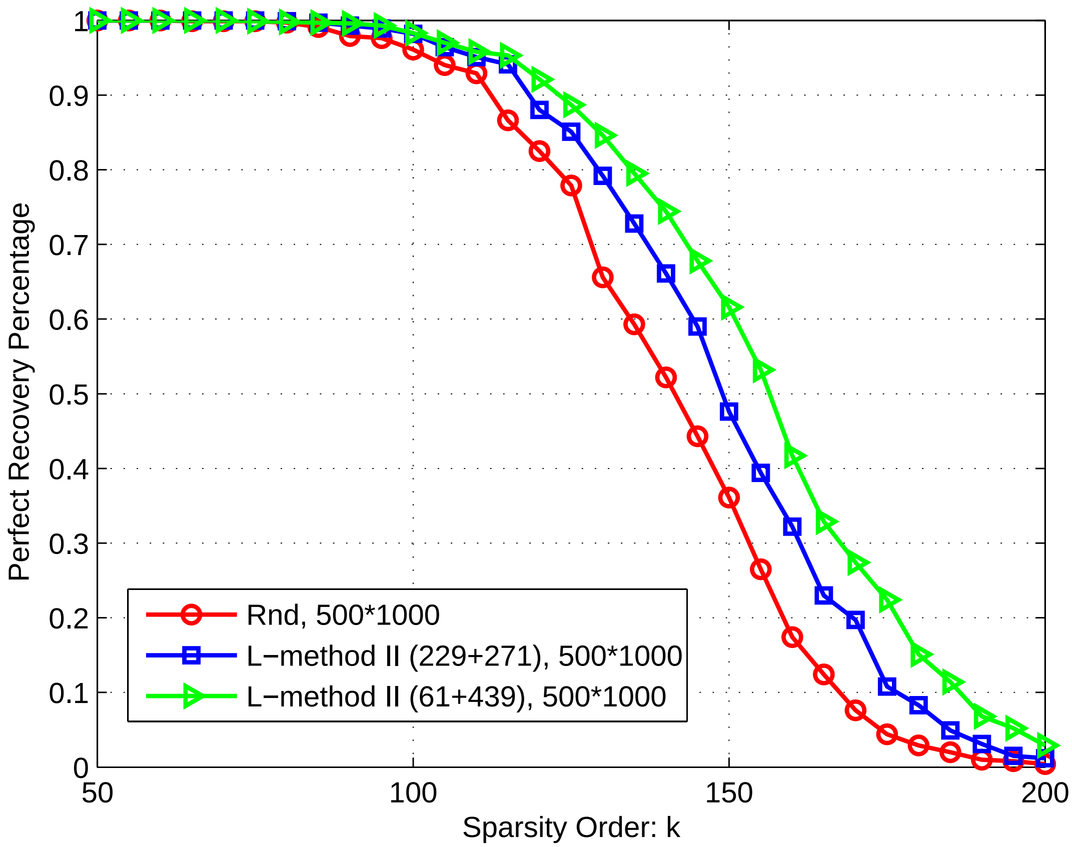

Fig. 3. Empirical performance of a measurement matrix from L-method II (difference splits) and the corresponding Gaussian matrix under OMP

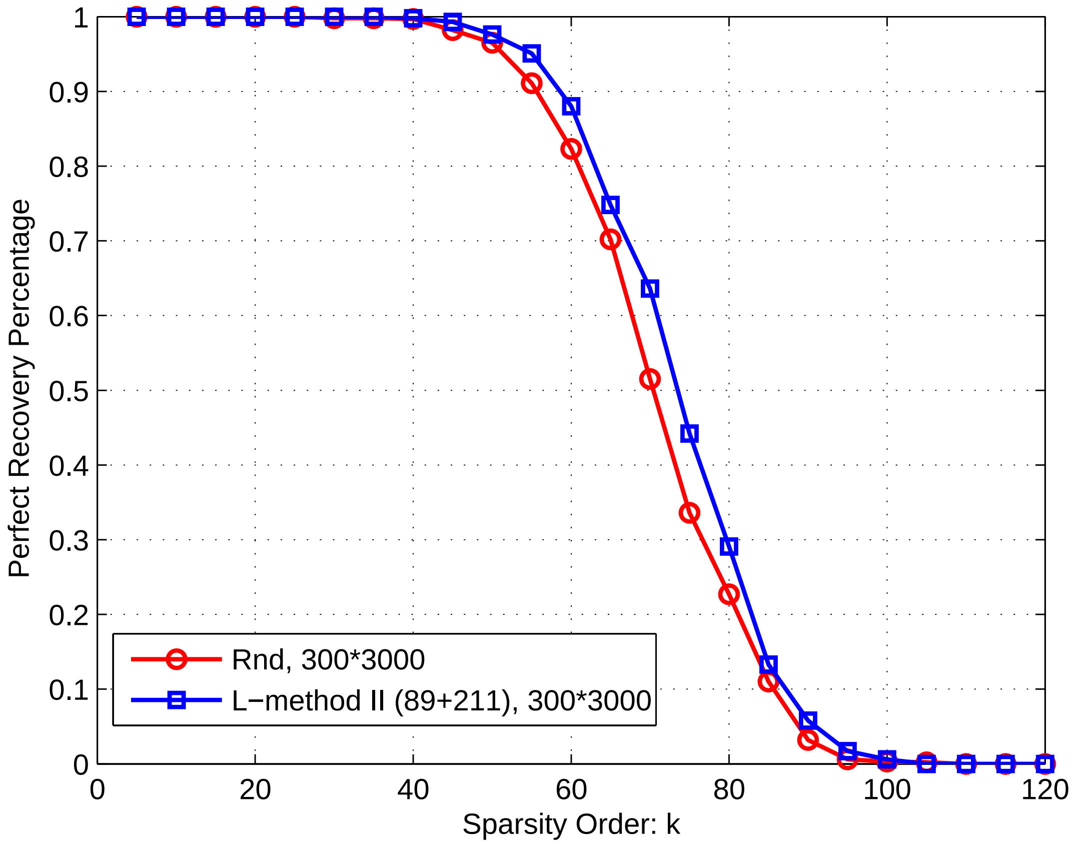

Fig. 4. Empirical performance of a measurement matrix from L-method II and the corresponding Gaussian matrix under OMP

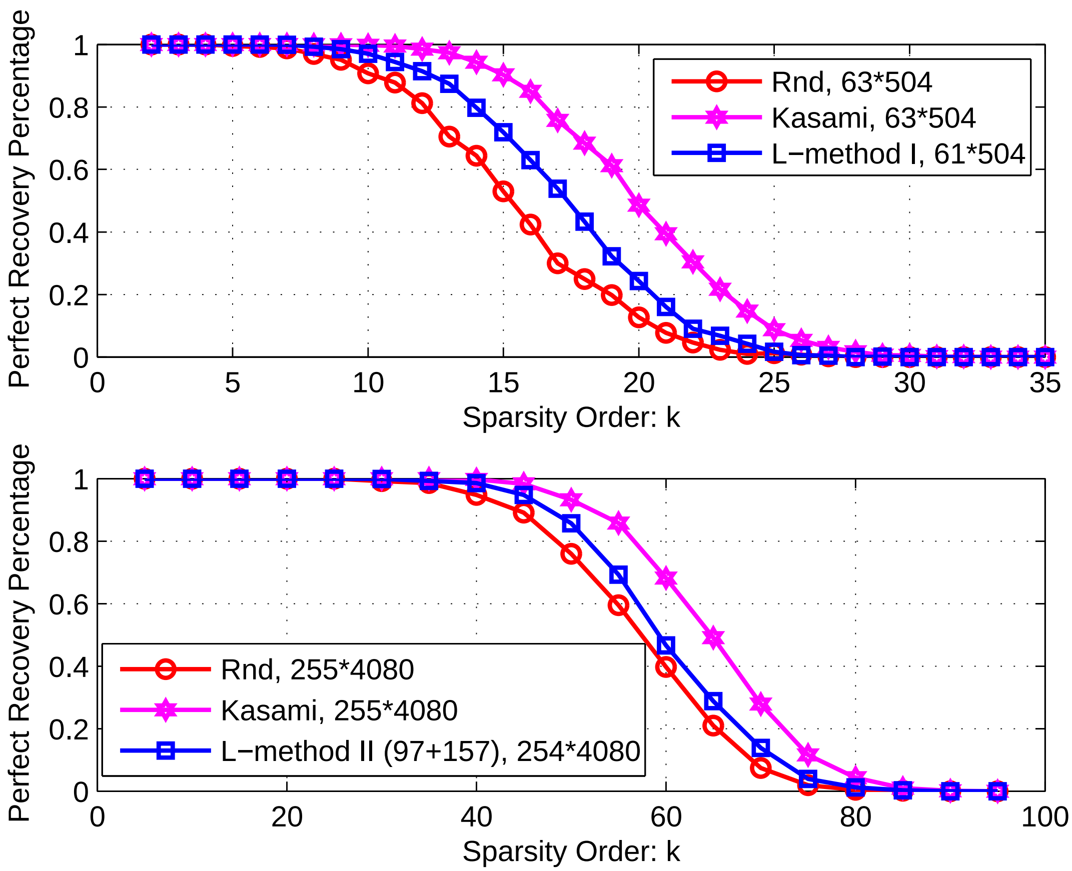

Fig. 5. Empirical performance of measurement matrices from L-method I/II and Kasami set (small), and the corresponding Gaussian matrix under OMP