Cyber-Physical Systems: Foundations, Principles and Applications (Elsevier)

Adaptive Control in Cyber-Physical Systems: Distributed Consensus Control for Wireless Cyber-Physical Systems

Guido Dartmann[1,2], Elham Almodaresi, Mahdi Barhoush[2], Naim Bajcinca[3], Gunes Karabulut Kurt[4], Volker Lücken[2], Ehsan Zhandi[2], Gerd Ascheid[2]

[1] University of Applied Sciences Trier, Germany.

[2] RWTH Aachen University, Germany.

[3] Technical University Kaiserslautern, Germany.

[4] Istanbul Technical University, Turkey.

Keywords: consensus, quantization, distributed control, multi-agent systems, protocol, cyber-physical systems

Abstract: In contrast to cable-bound connections, the wireless channel is an unreliable medium. Moreover, in distributed control, where multiple agents interact, a low latency in communication is important for a fast convergence of the distributed control algorithms.

In this chapter, we present a tutorial of the theory of consensus in multi-agent systems targeting wireless cyber-physical systems. Particularly, we address communication engineers. Additionally, we present how consensus can be achieved with a given communication topology and we discuss how a limited communication capacity on different communication links influences the consensus of such cyber-physical systems. As the quality of the wireless channel strongly influences the convergence of distributed control algorithms, we also present medium access control layer scheduling algorithms to reduce the latency and increase robustness in case of weak channel conditions.

1 Introduction

Distributed control and consensus are popular concepts in cyber-physical systems (CPS). Such systems can be further seen as a system of systems with multiple agents. An agent is here an independent sub-system (e.g., a robot). All agents work together to achieve a global utility. A typical application of a multi-agent system is depicted in Figure 1. This figure shows four unmanned flying vehicles (quadro-copters) communicating over wireless communication channels. The global goal is to reach a specific formation. This can be achieved, e.g., by an individual position and velocity estimate of each agent. The velocity and the current position are the states of each agent. To keep the formation of the entire group, each agent has to communicate its current state to its neighbors.

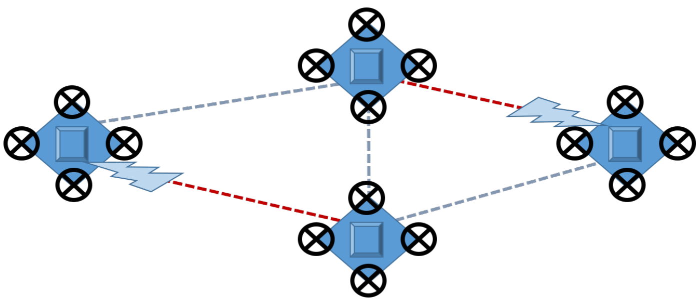


Figure 1: Example of a multi-agent system. Formation control is a typical application where consensus is of central importance, [Li 2015].

The consensus problem depends on the communication topology of the underlying multi-agent system. The topology is typically described by a so-called communication graph. In this graph, each vertex corresponds to an agent and each edge corresponds to a communication link. In this chapter, the communication link is considered to be a wireless channel and, therefore, may not always provide reliable communication. In control technology, a reliable communication with low latency is a fundamental requirement of a control system. In this chapter, we will give a tutorial on consensus of multi-agent systems. We study the consensus problem with wireless communication links and we investigate the influence of wireless communication parameters, such as signal-to-noise ratio (SNR) and channel fading on convergence of the system. Additionally, we observe, that latency is not only influenced by the mathematical consensus itself, it is also influenced by the communication model.

The chapter is organized as follows. In Section 2, we present an introduction to wireless channel models for cyber-physical systems. In Section 3.1, we present the main findings in this field which are relevant for multi-agent systems. Multi-agent systems can be categorized in static networks without changes of the topology and networks with the switching topology. In Section 3.2, we present the fundamentals of communication protocols for static networks. These fundamentals are required to understand networks with switching topologies which are presented in Section 3.3. In Sections 4 and 5, we present new ideas for adaptive quantization for rate limited cyber-physical systems.

2 Communication Channel of Multi-Agent Systems

The communication between agents constitutes an important part of the distributed control mechanism required for CPS. Conventionally, the communication network that enables agents to interact (and hence enable information exchange) is modeled as a graph [Godsil 2001]. Thereby, both directed and undirected graphs are frequently utilized. Due to a possible asymmetry in the wireless links considering uplink and downlink communication channels, a directed graph is more realistic to model a wireless CPS. A directed graph $\mathcal{G}$ is defined by its vertices and edges, $(\mathcal{V}, \mathcal{E})$. The elements of the nonempty vertex set $\mathcal{V} = \{v_1, \dots, v_n\}$ represent the agents. A total number of n agents is considered. An edge, (v_i, v_j), for any $v_i, v_j \in \mathcal{V}$, represents the information exchange between two vertices, v_i and v_j, if the corresponding agents can physically communicate. The edge set, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$, is an ordered set of pairs of nodes. Self loops, (v_i, v_i), are frequently excluded. If $(v_i, v_j) \in \mathcal{E}$, v_j is a neighbor of v_i. The set of neighbors of the agent v_i, $\mathcal{N}_i$, and the cardinality of the $\mathcal{N}_i$ is called the degree of v_i. A vertex is referred to as an isolated vertex if its neighbor set is empty. A weighted graph is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathrm{A})$. Here, $\mathrm{A} = [\mathrm{a_{ij}}] \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix of $\mathcal{G}$. In a weighted graph, a weight is associated with every edge and represents the associated cost of the edge.

A path on $\mathcal{G}$ is defined as a finite sequence of edges which connect a sequence of vertices. When there is an edge from every vertex to every other vertex, the corresponding directed graph is complete. The graph $\mathcal{G}$ defines the topology of the communication network. The frequently used error-free transmission and fixed topology assumptions [Xiao 2005], [Ahlswede 2000], [Ngai 2004] are valid wired networks. However, these assumptions are overly simplistic for wireless networks.

In wireless networks, the information exchanges among vertices take place over the air. Due to the unguided transmission environment, the transmitted information signals are subject to the impairments of the wireless communication channel, and these impairments are modeled using three main components; the path loss, the small-scale fading (or multi-path fading) and the large-scale fading (or shadowing). The path loss corresponds to the change in the average received power level related to the distance between the two vertices. The small scale fading $\mathrm{g_{ij}}$ results from the multi-path channel between the neighbor vertices and represents rapid fluctuations in the received signal's quality. Finally, as the third factor, the large scale fading represents the variations in the local mean received power. These three factors significantly affect the transmission quality, and causes non-negligible error rates when transmitting information through an edge. Furthermore, the adjacency matrix becomes time varying due to the dynamic topology due to the impacts of the wireless communication channel. Consequently, with wireless channels the topology of the CPS changes rapidly [Bai 2008], [Bai 2001] which can be modeled by a communication graph with $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}, \mathbf{A}_k)$ with a switching signal $\mathrm{k} = \mathrm{f(t)}$.

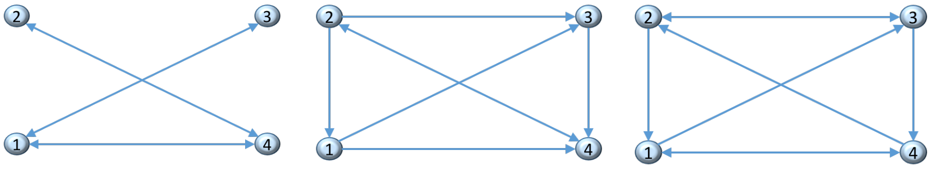
3 Consensus Control

In this section, we will present an overview of the mathematical fundamentals of consensus control. Consensus of a multi-agent system means the agreement of a common state of all agents and is mainly based on the topology of the communication network. The communication topology is mathematically described with so-called communication graphs. The mathematical theory to quantify the communication abilities of such multi-agent systems is based on algebraic graph theory.

3.1 Fundamentals of Algebraic Graph Theory

The fundamentals of algebraic graph theory were already developed in the early 1970s [Fiedler 1973]. Fiedler investigated the algebraic connectivity of graphs. Its seminal work is later used in the field of distributed control theory for multi-agent systems where communication graphs are used to model the communication among multiple agents. Fiedler's work is only valid for bidirectional communication which corresponds to undirected communication networks. In this case, it can be proved that a connected multi-agent system can achieve average-consensus. This is not given in case of directed graphs [Murray 2007]. Then, average consensus is only possible if the communication graphs are balanced.

Graph 1 undirected graph | Graph 2 directed graph and strongly connected | Graph 3 balanced directed graph



$$L(\mathcal{G}) = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix} \quad L(\mathcal{G}) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & -1 & 3 \end{bmatrix} \quad \mathbf{L}(\mathcal{G}) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Figure 2: Example for different types of connected multi-agent systems.

A typical example to understand algebraic graph theory is depicted in Figure 2. The figure presents three graphs with four nodes. We will use this example to explain the theory of this chapter. Each node or vertex v_i in the graph has an index $i \in \mathcal{I}$, where $\mathcal{I} = \{1,..n\}$ denotes the set of all indices which corresponds to the set of indices of all agents. Assuming all weights are $a_{ij} \in \{0,1\}$ have discrete values, the communication graph can be formally defined by the following three definitions [Fiedler 1973]:
Definition 1: Consider a $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{A})$ with n nodes. An edge e_{ij} has a non-zero weight a_{ij} if the nodes with index i and j are connected and $i \neq j$, otherwise it has the weight $a_{ij} = 0$ where $i, j \in \mathcal{I}$
Definition 2: The adjacency matrix $\boldsymbol{A}$, is $[\boldsymbol{A}]_{ij} = a_{ij}$, which denotes the weight of the edge from j to i.
Definition 3: The degree matrix $\boldsymbol{D}$ is a diagonal matrix with $[\boldsymbol{D}]_{ii} = d_{ii} = \sum_{j \neq i} a_{ij}$.
In Graph 1 of Figure 2, the weight of the edge between Node 1 and Node 3 is given $a_{13} = a_{31} = 1$ and the weight of the edge between Node 1 and Node 2 is given $a_{12} = a_{21} = 0$. Based on the definition of the graph, the so called Laplacian matrix can be associated with this graph [Fiedler 1973]:
Definition 4: The Laplacian matrix $\boldsymbol{L}(\mathcal{G}) \in \mathbb{R}^{n \times n}$ associated with $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{A})$ is $\boldsymbol{L}(\mathcal{G}) = \boldsymbol{D} - \boldsymbol{A}$.
The Laplacian matrix is essential to consensus control. An important parameter of this matrix is the set of eigenvalues. The matrix $\mathbf{L}(\mathcal{G})$ of an undirected graph is symmetric and positive semi-definite, therefore, also all eigenvalues are real non-negative. The second smallest eigenvalue λ_2 of the Laplacian matrix is called the algebraic connectivity of the graph and is often called Fiedler eigenvalue. In case of undirected graphs, the second smallest eigenvalue is larger in case of highly connected graphs compared to sparse graphs [Olfati-Saber 2004]. We will consider this property later in Section 3.3 when we discuss the convergence rate of consensus with switching topologies. Furthermore, it can be proved that in the case of directed graphs the real part of the eigenvalues is non-negative, which follows directly from the following theorem.
Theorem 1: [Olfati-Saber 2004, Theorem 1] Let $\eta(\mathcal{G})$ the maximum node degree of a directed communication graph with $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, then all eigenvalues of $L(\mathcal{G})$ are within the following set:

$$D(\mathcal{G}) = \{z \in \mathbb{C} : |z - \eta(\mathcal{G})| \leq \eta(\mathcal{G})\} \quad (1)$$

The set of possible locations of the complex eigenvalues is depicted in Figure 3.
Due to the construction of the Laplacian matrix we have zero row sums, $\sum_j l_{ij} = 0$, hence, $\boldsymbol{L}(\mathcal{G})\mathbf{1} = \mathbf{0}$ which implies that there is at least one eigenvalue $\lambda_1 = 0$.
All other eigenvalues of the symmetric Laplacian matrix satisfy: $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

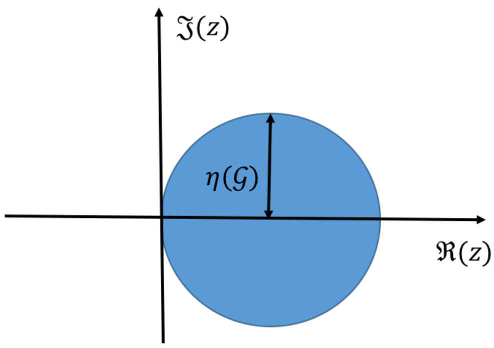


Figure 3: Complex plane where all eigenvalues of $\boldsymbol{L}(\mathcal{G})$ are located

The following graph definitions are important to understand the main theorems for consensus in multi-agent systems. In directed communication graphs, consensus is mainly associated with balanced and strongly connected graphs.
Definition 5: [West 2001] [Wolfram] A strongly connected digraph is a graph in which it is possible to reach any node starting from any other node by traversing edges in the direction(s) in which they point.
In the example of Figure 2, the Graph 1, 2, 3 are strongly connected,
Definition 6 : A graph is balanced if $\sum_j a_{ij} = \sum_j a_{ji}$ for all $i \in \mathcal{I}$.
From our example of Figure 2, we can see that Graph 1 and Graph 3 are balanced and we can also observe that undirected graphs are always balanced.
Proposition 1: [Oifati-Saber 2004, Theorem 1]. If $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{A})$ is strongly connected**,** then,
$rank\big(\boldsymbol{L}(\mathcal{G})\big) = n - 1.$
The following condition associated with a spanning tree. Note that, a strongly connect graph has a spanning tree.
Definition 7 : A spanning tree of graph is a tree which contains all nodes of the graph.
Proposition 2: e.g. [Ren 2008] Zero is a simple eigenvalue of $\boldsymbol{L}(\mathcal{G})$ if and only if the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{A})$ has a directed spanning tree.
Later in Section 3.2, we will see that the spanning tree inside a communication graph is an important property for consensus in multi-agent systems.
For further details of algebraic graph theory we refer to the seminal works of Fiedler [Fiedler 1979] and Wu [Wu 2005]. Fiedler contributed to this theory for weights $\mathrm{a_{ij}} \in \{0,1\}$. The theory also holds for generalized non-negative real valued $\mathrm{a_{ij}} \in \mathbb{R}^+$.

3.2 Consensus with Time-Invariant Topologies

In a multi-agent system, each node has individual states denoted by the state $\mathrm{x_i(t)} \in \mathbb{R}$. In this section, we will consider only scalar states. A generalization to multi-dimensional state vectors is presented in [Li, 2010] [Li, 2015].
Roughly speaking: The nodes of a multi-agent network are said to reach consensus if $x_i = x_j$ for all nodes $i, j \in \mathcal{I}$, [Olfati-Saber 2004]. For example: The communication of the Graph 2 in Figure 2 is as follows: In each time instant t, Node 1 forwards its state $x_1(t)$ to Node 4 and 3 and Node 4 forwards its states $x_4(t)$ to Node 2. If for $t \to \infty$: $x_1(t) = x_2(t) = x_3(t) = x_4(t)$, we have consensus.
The system node in the scalar case is simply:

$$\dot{\mathrm{x}}_\mathrm{i}(\mathrm{t}) = \mathrm{u_i(t)}. \tag{2}$$

A protocol per agent achieving consensus in specific cases is given by:

$$\mathrm{u_i(t)} = \sum_\mathrm{j} \mathrm{a_{ij}(x_j - x_i)}. \tag{3}$$

The protocol can be presented with the Laplacian in the compact notation:

$$\dot{\mathbf{x}}(\mathrm{t}) = -\mathbf{L}(\mathcal{G}) \cdot \mathbf{x}(\mathrm{t}) \tag{4}$$

with $\mathbf{x}(\mathrm{t}) = [\mathrm{x_1(t)}, \ldots, \mathrm{x_n(t)}]^\mathrm{T}$. A formal definition of consensus is presented in [Olfati-Saber 2004]:

Definition 8: [Olfati-Saber 2004] Let $\mathcal{X}: \mathbb{R}^n \to \mathbb{R}$, we say a protocol solves the $\mathcal{X}$-consensus problem if and only if there exists an asymptotically stable equilibrium $\boldsymbol{x}^*$satisfying $x_i^* = \mathcal{X}(\boldsymbol{x}(0))$.
Intuitively, consensus is reached if every stationary point satisfies $\dot{\mathbf{x}}(\mathrm{t}) = \mathbf{0}$. We have $\dot{\mathbf{x}}(\mathrm{t}) = \mathbf{0}$ if, e.g., $\mathbf{x}(\mathrm{t}) = \mathbf{1}$.The condition of $\mathbf{L}(\mathcal{G})\mathbf{1} = \mathbf{0}$ of the Laplacian matrix implies that $\sum_{\mathrm{i}=1}^{\mathrm{n}} \dot{\mathrm{x}}_{\mathrm{i}}(\mathrm{t}) = 0$ holds at consensus [Spanos 2005]. If consensus is achieved, we have $\mathbf{x}(\mathrm{t}) = \alpha \cdot \mathbf{1}$, therefore, we have

$$\dot{\mathbf{x}}(\mathrm{t}) = -\mathbf{L}(\mathcal{G}) \cdot \mathbf{x}(\mathrm{t}) = -\mathbf{L}(\mathcal{G}) \cdot \mathbf{1} \cdot \alpha = \mathbf{0}. \tag{5}$$

Theorem 2: [Bread 2003] The multi-agent network with protocol (4) achieves consensus if and only if the associated communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{A})$ has a spanning tree.
Hence, consensus is related to the existence of a spanning tree inside the communication graph.
If the communication graph is strongly connected, $\mathbf{L}(\mathcal{G})$ has a simple zero eigenvalue, then due to the property of Theorem 1, $-\mathbf{L}(\mathcal{G})$ has eigenvalues with non-positive real parts, therefore, the linear system (4 *)* is stable.
A special consensus case is the average-consensus which is defined by $\mathcal{X}(\mathbf{x}) = {}^{1}/_{\mathrm{n}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}}(0)$. Hence, in average-consensus the states converge to the average value of the initial states.
The main result concerning average-consensus in the literature is presented in the seminal work of [Olfati-Saber 2004]. The authors showed that average-consensus of a multi-agent system is given if the directed graph is strongly connected and if it is balanced [Ren, 2007]. In this case, $\mathbf{1}$ is the left eigenvector associated with the zero eigenvalue. Olfati-Saber et al. [Olfati-Saber 2004] proved:
Theorem 3: A strongly connected multi-agent network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{A})$ with protocol (4) achieves average-consensus if and only if $\mathbf{1}^{\boldsymbol{T}}\boldsymbol{L}(\mathcal{G}) = \mathbf{0}$.
In addition, the following holds also true:
Theorem 4: [Olfati-Saber 2004] $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{A})$ is balanced $\Leftrightarrow \mathbf{1}^{\boldsymbol{T}}\boldsymbol{L}(\mathcal{G}) = \mathbf{0} \Leftrightarrow \sum_{i=1}^{n} \dot{x}_i(t) = 0$.

3.3 Consensus with Switching Topologies

In practice, static topologies are not always feasible. Especially, in wirelessly linked CPS, effects like, path-loss, shadow-fading, and multi-path fading will result in disconnected links within the communication graph.
In specific low fading wireless channels, where no spanning tree exists in the communication graph, consensus is not possible. The consensus protocol of dynamic networks with switching topologies is similar to the consensus protocol for static multi-agent networks. The only difference is that the communication graph $\mathcal{G}_{\mathrm{k}} = (\mathcal{V}, \mathcal{E}, \mathbf{A}_{\mathrm{k}})$ has a time-variant adjacency matrix $\mathbf{A}_{\mathrm{k}}$ with $\mathrm{k} = \mathrm{f}(\mathrm{t})$ where within a fixed time interval, $\mathrm{f}(\mathrm{t})$ switches finitely many times. The function $\mathrm{f}: \mathbb{R}^+ \longrightarrow \{1, \dots, \mathrm{K}\}$ is a switching signal which indicates the changed topology of the communication graph. Let $\mathrm{t}_0, \mathrm{t}_1, \dots$ be a time sequence where $\mathrm{f}(\mathrm{t})$ switches. The function $\mathrm{f}(\mathrm{t})$ is a piecewise continuous function. The new protocol is, therefore, [Olfati-Saber 2004]:

$$\dot{\mathbf{x}}(\mathrm{t}) = -\mathbf{L}(\mathcal{G}_{\mathrm{k}}) \cdot \mathbf{x}(\mathrm{t}). \tag{6}$$

In case of static, strongly connected and balanced digraphs, average-consensus can be achieved. However, for arbitrary digraphs consensus is not always achievable.
Ren et al. [Ren, 2005] proved a similar relation between consensus and the spanning tree within the communication graphs as in the static case presented in Theorem 2. They used a union of directed graphs defined and proved that consensus can will be achieved asymptotically if union of graphs has a spanning tree.
Olfati-Saber et al. proved average-consensus for multi-agent systems with switching topologies. They used a disagreement value to prove their main result. This mathematical tool is also used in our contribution in Section 4. In case of an agreement (consensus), we have $\mathbf{x} = \alpha \cdot \mathbf{1}$. Therefore, we can define our disagreement state as follows [Olfati-Saber 2004]:

$$\mathbf{x} = \alpha \cdot \mathbf{1} + \boldsymbol{\delta}, \tag{7}$$

where $\boldsymbol{\delta}$ denotes the disagreement vector. Based on the definition in (7), we can also define the state space representation of the disagreement vector (disagreement dynamics)

$$\dot{\boldsymbol{\delta}} = -\mathbf{L}(\mathcal{G}_{\mathrm{k}}) \cdot \boldsymbol{\delta}. \tag{8}$$

4 Interaction of Control Theory and Information Theory

The interaction between control theory (consensus) and information theory is an important research field for the understanding of CPS. Due to limited data rate on communication links between two nodes, or unreliable communication links, the theory of static consensus as presented in Section 3 must be extended.

In the literature, disturbed states or measurements are often considered in the field of sensor networks [Xiao 2005], [Xiao 2007] or in consensus control with quantized states [Carli 2008], [Frasca 2000]:

$$u_i(t) = \sum_j a_{ij}\big(Q(x_j) - Q(x_i)\big). \tag{9}$$

The function $Q(x_j)$ denotes the quantized state. In [Dimarogonas 2010], [Guo 2013], the authors investigated a quantization protocol where the difference itself is quantized:

$$u_i(t) = \sum_j a_{ij} Q\big(x_j - x_i\big). \tag{10}$$

The authors of [Dimarogonas 2010] proved that the states converge to a ball with a bounded radius in case of a uniform quantization and the states converge to an equilibrium in case of logarithmic quantization. The authors of [Bauso 2009] considered disturbed measures $\tilde{Q}(x_j)$ only for the node j. Hence, they consider the following protocol:

$$u_i(t) = \sum_j a_{ij}\big(\tilde{Q}(x_j) - x_i\big). \tag{11}$$

Note that $\tilde{Q}$ is in [Bauso 2009] not a quantization function. In their work, the disturbance is $\tilde{Q}(x_j) = x_j + d_j$ where d_j denotes an unknown but bounded disturbance. Hence, it can be also seen as quantization error. The authors proved that under specific conditions defined in [Bauso 2009] an ε-consensus exists.

In [Li 2011], the authors made the first connection of consensus and communication theory. They investigated distributed average consensus with limited communication data rates and proved that a faster convergence requires a higher quantization.

In this section, we will start with Shannon's channel capacity and we will directly link the signal-to-noise ratio (SNR) of a channel to the quantization of a link in a communication graph. We will prove that for a single time step the quantization error is bounded by the SNR of the channel. Each link between a node i, and a node j has a specific rate $R_{ij} = \log(1 + \gamma_{ij})$, with the instantaneous SNR $\gamma_{i,j} = |g_{ij}|^2 \cdot \mathrm{SNR}$ which is given by the upper bound in form of the channel capacity on each link. The capacity of the channel is given by Shannon's famous theorem:

Theorem 5: [Cover 2005] The capacity of a Gaussian channel with an SNR $\gamma_{ij} = g_{ij} \cdot SNR$ is given by:

$$C_{ij} = \log\big(1 + \gamma_{ij}\big). \tag{12}$$

The available rate is upper bounded by the channel capacity $R_{ij} = C_{ij}$. The maximal quantization error for a given data rate R_{ij} is given by: $e_j = c/2^{R_{ij}+1}$, where $c = \max(x_j) - \min(x_j)$. Hence, we can rephrase the quantization error in terms of the current SNR

Proposition 3: The worst case quantization error is given by:

$$e_j = \frac{c}{2(1 + \gamma_{ij})}. \tag{13}$$

Proof: The proof follows from Theorem 5 and $e_j = c/2^{N_{Bits}+1}$, hence, $\frac{c}{2e_j} = 2^{N_{Bits}} \Leftrightarrow 1 + \gamma_{ij} = \frac{c}{2e_j}$.

In what follows, we consider a strongly connected and balanced topology. Furthermore, we only consider Gaussian noise, then the SNR will be: $\mathrm{SNR} = P/\sigma_n^2$ where P denotes the power of the useful signal and σ_n^2 denotes the variance of the noise. Let us further define the worst case quantization error for additive white Gaussian noise (AWGN) channels without channel fading ($|g_{ij}|^2 = 1$) as

$$\varepsilon(\mathrm{SNR}) = \frac{c}{2(1 + \mathrm{SNR})}. \tag{14}$$

The following protocol is defined for a communication system where the nodes are transmitting quantized values:

$$u_i(t) = \sum_j a_{ij}\big(Q_j(t) - Q_i(t)\big) = \sum_j a_{ij}\big(x_j(t) + e_j(t) - x_i(t) - e_i(t)\big) \tag{15}$$

where, $Q_i(t)$ and $Q_j(t)$ are the quantized values of $x_i(t)$ and $x_j(t)$. Note, that this protocol is a generalization of the following protocol.

If $Q_i(t)$ is a perfect quantization the error $e_i(t) = 0$. In our communication model we will have a limited capacity form node i to node j and perfect knowledge of the own state $x_i(t)$. In what follows, we use undirected graphs $\mathcal{G}$ to simplify the derivations. Equation (15) is written as:

$$\mathbf{u}(t) = -\mathbf{L}\big(\mathbf{x}(t) + \mathbf{e}(t)\big)\,, \tag{16}$$

where, $\mathbf{e} = [e_1, \dots, e_n]^T$. It can be simply checked that $\mathbf{1}^T\mathbf{u}(t) = 0$. Then also $\mathbf{1}^T\dot{\mathbf{x}}(t) = 0$ holds. This means $\mathbf{x}(t)$ is a constant over time. Subsequently, $\mathbf{x}(t)$ can be decomposed to two parts $\alpha = \frac{1}{n}\,\mathbf{1}^T\mathbf{x}(0)$ which is the constant mean value of $\mathbf{x}(t)$ over the time, and a deviation vector $\boldsymbol{\delta}(t)$. Hence,

$$\mathbf{x}(t) = \alpha\mathbf{1} + \boldsymbol{\delta}(t)\,. \tag{17}$$

Now, we will prove that $\mathbf{x}(t)$ converges to $\alpha\mathbf{1}$, and $\boldsymbol{\delta}(t)$ is bounded for high values of time. The following proof lines are borrowed from [Seyboth 2013]. From (16) it follows,

$$\dot{\boldsymbol{\delta}}(t) = -\mathbf{L}(\mathcal{G})\big(\boldsymbol{\delta}(t) + \mathbf{e}(t)\big) \tag{18}$$

Hence,

$$\boldsymbol{\delta}(t) = e^{-\mathbf{L}(\mathcal{G})t}\,\boldsymbol{\delta}(0) + \int_0^t e^{-\mathbf{L}(\mathcal{G})(t-\tau)}\mathbf{L}(\mathcal{G}) \cdot \mathbf{e}(\tau)d\tau \tag{19}$$

Then, the deviation vector $\boldsymbol{\delta}(t)$ satisfies:

$$\|\boldsymbol{\delta}(t)\| \le \left\|e^{-\mathbf{L}(\mathcal{G})t}\,\boldsymbol{\delta}(0)\right\| + \int_0^t \left\|e^{-\mathbf{L}(t-\tau)}\mathbf{L}(\mathcal{G}) \cdot \mathbf{e}(\tau)\right\|d\tau \tag{20}$$

We have $\left\|e^{-\mathbf{L}(\mathcal{G})t}\,\boldsymbol{\delta}(0)\right\| \le e^{-\lambda_2 t}\|\,\boldsymbol{\delta}(0)\|$ [Ofati Saber 2004], [Seyboth 2013] and since the Laplacian $\mathbf{L}(\mathcal{G})$ is a Laplacian of an undirected, connected graph $\mathcal{G}$, therefore, $\mathcal{G}$ is also balanced we have $\mathbf{1}^T\mathbf{L}(\mathcal{G})\mathbf{e}(\tau) = 0$. Consequently, with [Seyboth 2011, Lemma 1] we have:

$$\|\boldsymbol{\delta}(t)\| \le e^{-\lambda_2 t}\|\,\boldsymbol{\delta}(0)\| + \int_0^t e^{-\lambda_2(t-\tau)}\|\mathbf{L}(\mathcal{G}) \cdot \mathbf{e}(\tau)\|d\tau \tag{21}$$

Where, $\lambda_2(G)$ is the algebraic connectivity of the graph. Also, $\|\mathbf{e}(\tau)\| \le \Delta(\mathrm{SNR}) \le \sqrt{n} \cdot \varepsilon(\mathrm{SNR})$, where $\Delta(\mathrm{SNR})$ is the maximum quantization error, and $\|\mathbf{L}(\mathcal{G}) \cdot \mathbf{e}(\tau)\| \le \|\mathbf{L}(\mathcal{G})\| \cdot \|\mathbf{e}(\tau)\|$. Then,

$$\|\boldsymbol{\delta}(t)\| \le e^{-\lambda_2 t}\|\,\boldsymbol{\delta}(0)\| + \int_0^t e^{-\lambda_2(t-\tau)}\|\mathbf{L}(\mathcal{G})\|\Delta(\mathrm{SNR})d\tau \tag{22}$$

Hence,

$$\|\boldsymbol{\delta}(t)\| \le e^{-\lambda_2 t}\|\,\boldsymbol{\delta}(0)\| - \frac{\Delta(\mathrm{SNR})\|\mathbf{L}(\mathcal{G})\|}{\lambda_2}\ e^{-\lambda_2 t} + \frac{\Delta(\mathrm{SNR})\|\mathbf{L}(\mathcal{G})\|}{\lambda_2}, \tag{23}$$

which means that $\|\boldsymbol{\delta}(t)\|$ converges to $\frac{\Delta(\mathrm{SNR})\|\mathbf{L}(\mathcal{G})\|}{\lambda_2}$. This indicates smaller deviation from the consensus value α for smaller values of the quantization error. We can summarize the previous derivation in the following theorem:

Theorem 6: Let $\mathcal{G}$ be a static undirected graph, the disagreement vector is bounded as follows:

$$\|\boldsymbol{\delta}(\infty)\| \le \frac{\Delta(\mathrm{SNR})\|\mathbf{L}(\mathcal{G})\|}{\lambda_2}. \tag{24}$$

If the $\Delta(\mathrm{SNR})$ decreases also the error decreases. The lower the SNR in a time instant, the larger the quantization error. Hence, to control the system we need an adaptation of the quantization error. Concepts for this adaptation are introduced in the next section.

5 Cross-Layer Design Resource Allocation for Distributed Control

In this section, we present transmission protocols to coordinate the transmission of the nodes in case of a wireless channel as a shared medium. Here, we only consider so-called time division multiple access (TDMA) schemes which means that the channel access is coordinated among different time slots. Due to the limited bandwidth of the shared wireless channel medium, the quantization, which can be used to transmit the state to another user, is limited. Hence, there will be a quantization error as introduced in Section 4.

Communication is structured in layers [Tanenbaum 2003]. Here we use a simplified structure depicted in Figure 4. The channel ressources are also called physical (PHY) layer resource. Here, we consider only time slots which must be shared by different transmitters. The medium access (MAC) – PHY layer

coordinates the allocation of resources. Above this layer, there is an interface to the application (APP) layer. The MAC-APP layer adapts the quantization based on the available resources in the MAC-PHY layer. The communication from node i to node j is, therefore, as follows: node i knows the current channel state (proportional to SNR) and, therefore, the available resources. Then, node i can, e.g., choose an adapted quantization to discretize its state which has to be transmitted to node j. The MAC-PHY layer assigns resources for the transmission of the quantized state and encodes it for the transmission over the PHY layer by choosing a sufficient channel coding and modulation scheme available for the current SNR.
In the following subsections, we will present two simple MAC layer protocols for the adaptation of the quantization or the transmission period. The first scheme is called adaptive quantization (AQ). Here the transmission delay is constant, however, the quantization is adaptive. The second scheme is called adaptive transmission length (ATL). In the ATL scheme, the quantization is constant but the transmission duration is variable.

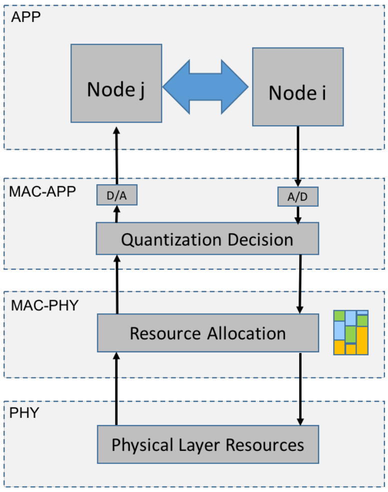


Figure 4: Communication structure of the consensus system

5.1 Adaptive Quantization (AQ)

The idea is very similar to the discussion in Section 4. The transmitted variables are sent via wireless channels with limited capacity. Here, we assume a fading channel with an SNR given by $\gamma_{ij} = g_{ij} \cdot \mathrm{SNR}$ between two nodes i, j. The variable g_{ij} denotes the channel gain. The channel gain is constant during a slot of length T_s. In each time slot we have a different SNR. The transmit scheme is time division multiple access (TDMA). Each transmitter has a specific transmit period $T_P = T_s$ which is equal to the slot length. After this transmit period the next transmitter is scheduled. Hence, the shared channel is divided into orthogonal time slots.

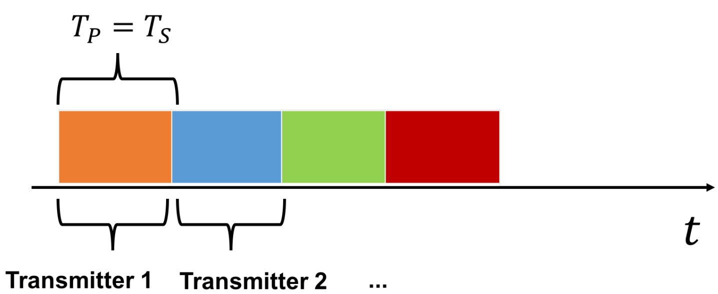


Figure 5: TDMA transmission

In each slot the transmitter can use the bandwidth B. The upper bound of the achievable rate to transmit a specific state from i to j within this slot period is given by:

$$R_{ij}(k) = B \cdot \log_2 \left(1 + \mathrm{SNR} \cdot \left|g_{ij}(k)\right|^2\right) \ \text{bits/s}. \tag{25}$$

The transmitter can estimate the SNR of the next slot, e.g., based on the uplink signal. The rate $R_{i,j}$ is variable and depends on SNR of each link. Therefore, the number of bits that can be transmitted during a transmission period T_P is changing. This results in the following protocol:

Protocol 1: AQ. The number bits determines the quantization which can be used for the quantization of the state which will be transmitted from node i to node j.

To discretize the system, we use the same discrete time model as presented in [Olfati-Saber 2004] with a step-size $\varepsilon_d > 0$, hence, the update (2) will be:

$$x_i(k+1) = x_i(k) + \varepsilon_d \cdot u_i(k) \tag{26}$$

The quantization $Q(x_j)$ of the state x_j is differential, hence, the node j transmits the quantized difference $Q\left(x_j(k) - \hat{x}_j(k-1)\right)$. We furthermore assume that the own state $x_i(t)$ is perfectly known $e_i(t) = 0$. The transmitter knows, the quantization, therefore, the transmitter also knows the signal $\hat{x}_j(k-1)$ updated by the receiving node. The receiving node i then updates its estimation of $\hat{x}_j$ by

$$\hat{x}_j(k) = \hat{x}_j(k-1) + Q\left(x_j(k) - \hat{x}_j(k-1)\right) = x_j(k) + \Delta_j(k). \tag{27}$$

Table 1 presents the first three steps of the proposed protocol with differential quantization.

Based on the estimated SNRs of all links, the transmitter chooses the minimum capacity (and according quantization) over active connections from the transmitting node to all receiving nodes.

The adaptive quantization protocol ensures a fixed transmission delay, however, the error due to the quantization in case of low SNR can be very large. In Figure 6, the quantization function $Q(x)$ for different SNR values is depicted. The larger the SNR, the better the resolution of the quantization function. To simplify the investigations we assume $B\,T_S = 1$ and $\left|g_{ij}(k)\right|^2 = 1$ for all i, j. In case of an SNR of just 5dB only 2 bits are available, because $log_2(1 + SNR) = 2.0574\, bits$. The initial states are given by $\boldsymbol{x}(0) = [-0.2,\ 0.1,\ 0.2, -0.1]^T$. The average value is $\alpha = 0$, hence, all states will converge to zero, in case average consensus is achievable.

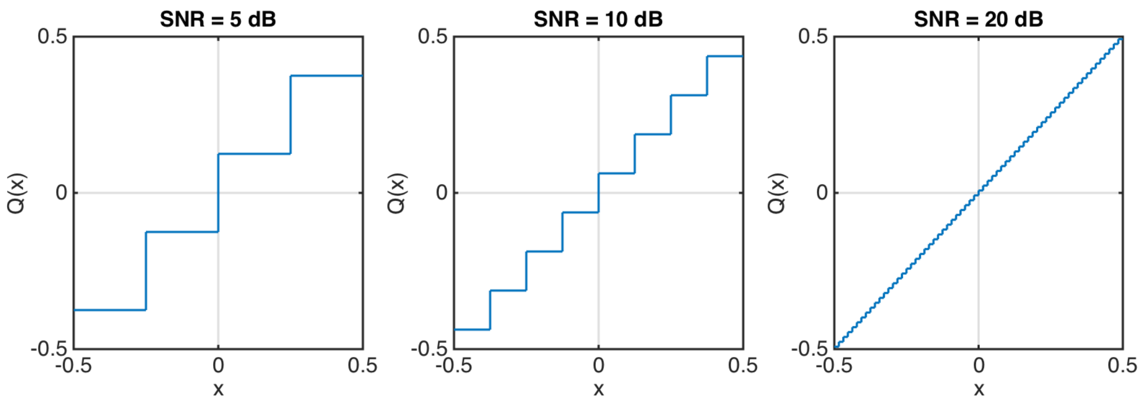


Figure 6: Quantization at different SNR values.

	Node j			Node i
Time	State	Sent to node i	Knowledge of its state at node i	Updated value of state j
0	$x_j(0) = \hat{x}_j(0)$	$x_j(0) = \hat{x}_j(0)$	$x_j(0) = \hat{x}_j(0)$	$x_j(0) = \hat{x}_j(0)$
1	$x_j(1)$	$Q\left(x_j(1) - \hat{x}_j(0)\right)$	$\hat{x}_j(1) = x_j(1) + \Delta_j(1)$	$\hat{x}_j(1) = \hat{x}_j(0) + Q\left(x_j(1) - \hat{x}_j(0)\right)$ $= x_j(1) + \Delta_j(1)$
2	$x_j(2)$	$Q\left(x_j(2) - \hat{x}_j(1)\right)$	$\hat{x}_j(2) = x_j(2) + \Delta_j(2)$	$\hat{x}_j(2) = \hat{x}_j(1) + Q\left(x_j(2) - \hat{x}_j(1)\right)$ $= x_j(2) + \Delta_j(2)$
3	$x_j(3)$	$Q\left(x_j(3) - \hat{x}_j(2)\right)$	$\hat{x}_j(3) = x_j(3) + \Delta_j(3)$	$\hat{x}_j(3) = \hat{x}_j(2) + Q\left(x_j(3) - \hat{x}_j(2)\right)$ $= x_j(3) + \Delta_j(3)$
…				

Table 1: First three steps of the AQ protocol with differential quantization.

Figure 7 presents the simulation results for a network with four nodes. The communication graph is a complete and undirected graph. We can observe that in case of a low quantization error, a final disagreement is the result, while in case of a high SNR, average consensus can be achieved.

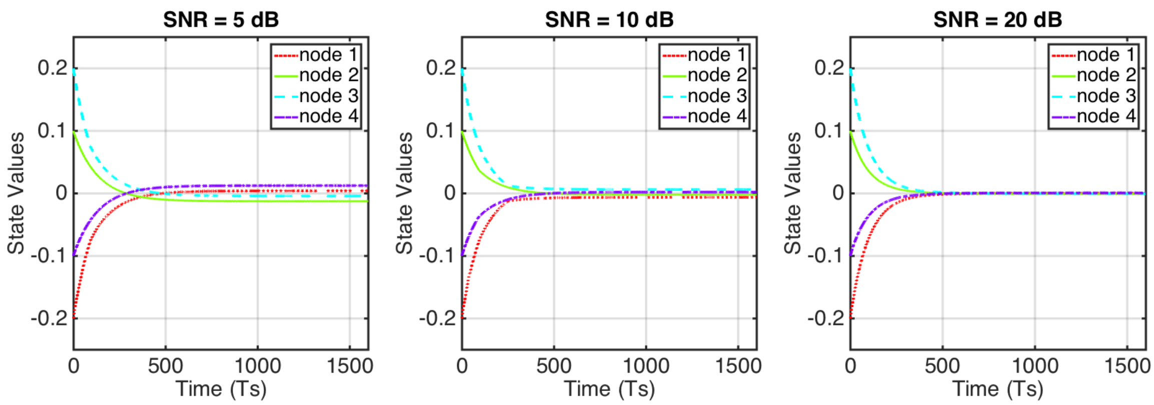


Figure 7: Simulation Results for AQ with a complete graph with four nodes and different SNR values.

5.2 Adaptive Transmission Length (ATL)

In this section, we assume a fading channel with an SNR given by $\gamma_{ij}(k) = g_{ij}(k) \cdot SNR(k)$ between two nodes i, j. The Variable $g_{ij}(k)$ denotes the channel gain. In each timeslot k we have a different SNR. The transmit scheme is also TDMA. However, now the quantization is fixed. Hence, the transmitter must use multiple slots in case the achievable rate within a single slot was not sufficient to transmit the state with the given quantization. Therefore, each transmitter has a transmit period which consists of multiple time slots $T_P = N \cdot T_s$. After this transmit period the next transmitter is scheduled. The upper bound of the achievable rate is then given by:

$$R_{ij} = \sum_{k=1}^{N} B \cdot log_2\left(1 + SNR \cdot \left|g_{ij}(k)\right|^2\right) \ bits \qquad (28)$$

The rate R_{ij} is fixed and determined by the used quantization. The protocol is defined as follows:

Protocol 2: ATL. The quantization used for the transmission of the states is fixed. To ensure a fixed quantization for the transmission of the state of a node i to node j, multiple slots must be used. The transmission is complete if all bits for the current quantized states are transmitted. In case of multiple active links from one transmitting node to multiple receiving nodes, the link with the lowest SNR determines the (largest) number of slots which is necessary to transmit the current state with the given quantization error free to all receivers.

Figure 8 presents the slot structure of the proposed protocol.

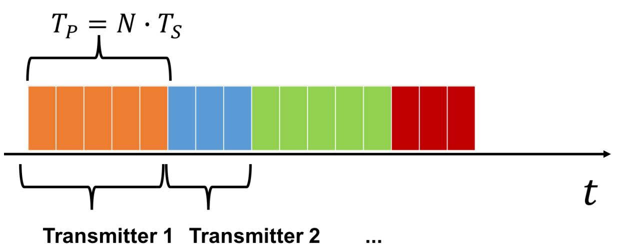


Figure 8: TDMA scheme used for the transmission. Here each transmitter has a different number of slots for the transmission of its current state.

In contrast to the AQ protocol, which has a variable quantization but guarantees a transmission delay due to the fixed transmission period T_P, we now have a variable transmission period $T_P = N \cdot T_s$ and a fixed quantization. The node will continue the transmission until all bits of the state with the given quantization are transmitted. This scheme needs an additional negotiation among the nodes due to the variable transmission periods.
Figure 9 presents the simulation results for ATL and the same stetting as in the previous section. Here, we use a desired quantization of 6 bits. To achieve the desired quantization a different number N of slots is required depending on the given SNR. The higher the SNR the faster is, therefore, the convergence. Compared to AQ, the final accuracy is the same for all SNR values. In contrast to ATL, AQ has the same convergence rate until the possible accuracy is reached. ATL has for all simulated SNRs the same accuracy, however, a different convergence time.

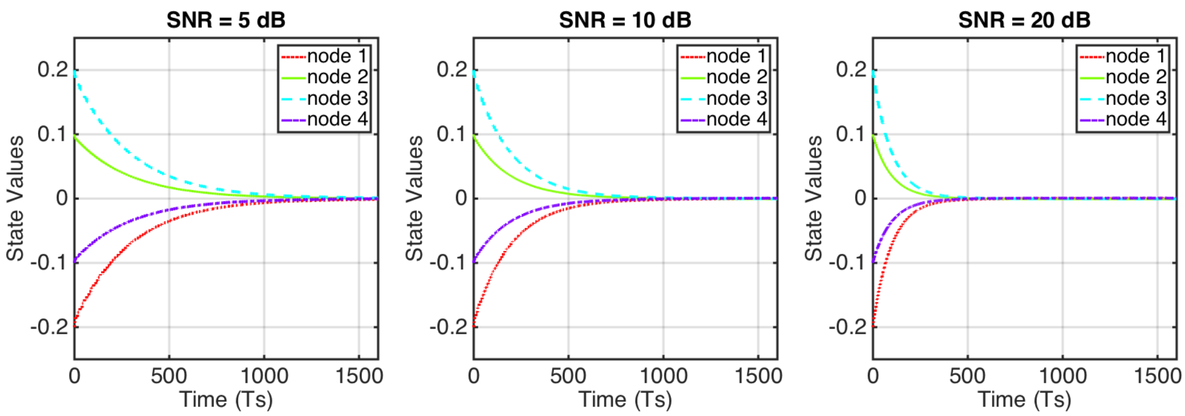


Figure 9: Simulation results for ATL with a complete graph and different SNR values.

6 Conclusions and Emerging Topics

This chapter presented how a limited capacity on the links results in an additive consensus error. This error depends on the current SNR of the communication links. Two communication protocols AQ and ATL are proposed for these capacity limited multi-agent systems. Comparing the results of AQ and ATL, we can observe that high SNR can achieve either a high accuracy (AQ) or a fast convergence (ATL) in case of a given, desired quantization.

In addition to the SNR, the network topology is a further performance parameter. Based on the discussion of the previous sections, we can observe that a dense topology is not always advantageous compared to a sparse topology. A dense topology can achieve a fast convergence in a mathematical sense of the consensus system. However, then multiple slots are required increasing the convergence time. Therefore, the future work must investigate the trade-off between consensus convergence and sparse allocation of communication resources. The topology must be optimized based on communication channel states. However, the communication channel can be influenced by the system dynamics. Therefore, a challenging research topic for future research is the joint control of the system dynamics with respect to the communication abilities resulting from a specific system topology.

7 References

Ahlswede 2000 — Ahlswede, R.; Cai, N. ; Li, S.-Y.; Yeung, R.; “Network information flow,” IEEE Trans. Inf. Theory, vol. 46, no. 4, , July 2000, pp. 1204–1216

Bai 2008 — Bai, B. ; Chen, W. ; Cao, Z.; Ben Letaief, K.; “Achieving high frequency diversity with subcarrier allocation in OFDMA systems,” in IEEE Global Telecommunications Conference (ICC), Nov 2008

Bai 2011 — Bai, B.; Chen, W.; Ben Letaief, K.; Cao, Z.; “Diversity-multiplexing tradeoff in OFDMA systems: An h-matching approach,” IEEE Transactions on Wireless Communications, vol. 10, no. 11, pp. 3675–3687, November 2011.

Bauso 2009 — Bauso, D.; Giarré, L.; Pesenti, R.; “Consensus for Networks with Unknown but Bounded Disturbances“ SIAM J. Control Optim., 48(3), 1756–1770, 2009

Beard 2003 — Beard, R. W., Stepanyan, V., "Synchronization of Information in Distributed Multiple Vehicle Coordinated Control, " IEEE Conference on Decision and Control, 2003

Carli 2008 — Carli, R.; Fagnani, F.; Frasca, P.; Zampieri, S., "The quantization error in the average consensus problem," Control and Automation, 2008 16th Mediterranean Conference on , vol., no., pp.1592,1597, 25-27 June 2008

Cover 2005 — Cover, T. M.; Thomas, J. A., "Elements of Information Theory, " Wiley Series in Telecommunications and Signal Processing, 2005

Dimarogonas 2010 — Dimarogonas, D. V.;, Johansson, K. H.; "Stability analysis for multi-agent systems using the incidence matrix: Quantized communication and formation control",Automatica, vol. 46, Issue 4, April, 2010

Fiedler 1973 — Fiedler, M.; "Algebraic connectivity of graphs," Czechoslovak Mathematical Journal (1973), vol: 23, Issue: 2, page 298-305

Frasca 2008 — Frasca, P.; Carli, R.; Fagnani, F.; Zampieri, S., "Average consensus on networks with quantized communication", Intern. Journ. on Non-linear and Robust Control, 2008

Godsil 2001 — Godsil, C.; Royle G. "Algebraic Graph Theory". New York: Springer, 2001.

Guo 2013 — Guo, M.; Dimarogonas, D. V.; "Consensus with Quantized Relative State Measurements", Automatica, 49(8): 2531-2537, Aug 2013.

Li 2011 — Li, T.; Fu, M.; Xie, L.; Zhang, J.-F., "Distributed Consensus With Limited Communication Data Rate," Automatic Control, IEEE Transactions on , vol.56, no.2, pp.279,292, Feb. 2011

Li 2010 — Li, Z.; Duan, Z.; Chen, G.; Huang, L., "Consensus of Multi-agent Systems and Synchronization of Complex Networks: A Unified Viewpoint," Circuits and Systems I: Regular Papers, IEEE Transactions on , vol.57, no.1, pp.213,224, Jan. 2010

Li 2015 — Li, Z.; Duan, Z, "Cooperative Control of Multi-Agent Systems: A Consensus Region Approach," CRC Press, Automation and Control Engineering, Oct. 2015

Ngai 2004 — Ngai, C. K.; Yeung, R.; “Network coding gain of combination networks,” in Proc. IEEE Inf. Theory Workshop, Oct. 2004, pp. 283–287.

Murray 2007 — Murray, R. M., "Recent Research in Cooperative Control of Multivehicle Systems Journal of Dynamic Systems," Measurement, and Control, vol. 129, no. 5., pp. 571-583, 2007

Olfati-Saber 2004 — Olfati-Saber, R.; Murray, R.M., "Consensus problems in networks of agents with switching topology and time-delays," Automatic Control, IEEE Transactions on, vol.49, no.9, pp.1520,1533, Sept. 2004

Olfati-Saber 2007 Olfati-Saber, R.; Fax, J.A.; Murray, R.M., "Consensus and Cooperation in Networked Multi-Agent Systems," Proceedings of the IEEE, vol.95, no.1, pp.215,233, Jan. 2007

Ren 2005 Ren, W.; Beard, R.W., "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," Automatic Control, IEEE Transactions on , vol.50, no.5, pp.655,661, May 2005

Ren 2007 Wei Ren; Beard, R.W.; Atkins, E.M., "Information consensus in multivehicle cooperative control," Control Systems, IEEE , vol.27, no.2, pp.71,82, April 2007

Ren 2008 Ren, W.; Beard, R. W., "Distributed Consensus in Multi-vehicle Cooperative Control Theory and Applications, "Communications and Control Engineering, Springer, 2008

Spanos 2005 Spanos, D. P.; Olfati-Saber, R.; Murray R. M. "Dynamic Consensus for Mobile Networks," IFAC World Congress, 2005

Seyboth 2011 Seyboth, G., Dimarogonas, D.V., Johansson, K.H., Control of multi-agent systems via event-based communication. In Proc. 18th IFAC World Congress, Milan, Italy 2011.

Seyboth 2013 Seyboth, G., Dimarogonas, D.V., & Johansson, K.H. Event-based broadcasting for multi-agent average consensus. Automatica,49(1):245-252, 2013.

Tanenbaum 2003 Tanenbaum, A. S. "Computer Networks," 4th Version. Englewood Cliffs, NJ: Prentice-Hall, 2003.

West 2000 West, D.B., "Introduction to Graph Theory," Prentice Hall, 2 Edition, Sep. 2000

Wolfram Online: www.wolfram.de Date: 4th January 2016, Link: http://mathworld.wolfram.com/StronglyConnectedDigraph.html

Wu 2005 Wu, C.W., "Algebraic connectivity of directed graphs," Linear and Multilinear Algebra, vol. 53, Issue 3, 2005

Xiao 2005 Xiao, L.; Boyd, S.; Lall, S., "A scheme for robust distributed sensor fusion based on average consensus," Information Processing in Sensor Networks, 2005. IPSN 2005. Fourth International Symposium on , vol., no., pp.63,70, 15 April 2005

Xiao 2007 Xiao, L.; Boyd, S.; Kim, S.-J., "Distributed Average Consensus with Least-Mean-Square Deviation," Journal of Parallel and Distributed Computing, 67(1):33-46, 2007.

Author Information:

Prof. Dr.-Ing. Guido Dartmann, University of Applied Sciences Trier, Environmental Campus Birkenfeld, Campusallee, 55768 Hoppstädten-Weiersbach, g.dartmann@umwelt-campus.de

Dr. Elham Almodaresi, Yazd University, Department of Mechanical Engineering, Iran, P.O. Box: 89195-741, Postal code: 891686951, Yazd, Iran, e.modarresi@gmail.com

Mahdi Barhous, Msc., RWTH Aachen University, Institute for Communication Technologies and Embedded Systems, Templergraben 55, 52062 Aachen, mahdi.barhous@ice.rwth-aachen.de

Prof. Dr.-Ing. Naim Bajcinca, University of Kaiserslautern, Gottlieb-Daimler-Str. Geb. 42, 67663 Kaiserslautern, Germany, naim.bajcinca@mv.uni-kl.de

Prof. Dr. Gunes Karablulut Kurt, Istanbul Technical University, Faculty of Electrical and Electronics Engineering Maslak 34469, Istanbul, Turkey, gkurt@itu.edu.tr

Dipl.-Ing. Volker Lücken, RWTH Aachen University, Institute for Communication Technologies and Embedded Systems, Templergraben 55, 52062 Aachen, volker.luecken@ice.rwth-aachen.de

Ehsan Zandi, Msc. RWTH Aachen University, Chair for Theoretical Information Technology, Templergraben 55, 52062 Aachen, zandi@ti.rwth-aachen.de

Prof. Dr.-Ing. Gerd Ascheid, RWTH Aachen University, Institute for Communication Technologies and Embedded Systems, Templergraben 55, 52062 Aachen, gerd.ascheid@ice.rwth-aachen.de