Analyzing Routing Strategy NFP in Multihop Packet Radio Networks on a Line

Rudolf Mathar and Jurgen Mattfeldt

Abstract — **We present a one-dimensional model to analyze routing strategy NFP (Nearest with Forward Progress) for a multihop packet radio network. It is assumed that each station has adjustable transmission range to address target nodes, distributed on a line according to a nonhomogeneous Poisson process. NFP transmits to the nearest neighbor in the desired direction with transmission range as small as possible, to minimize the probability of collisions. This model is appropriate e.g. for road traffic information systems. Our analysis is based on a complete mathematical description, the solution of certain differential equations is one of the key points to arrive at a closed form solution. Results are presented graphically. It turns out that NFP has uniformly largest throughput, while progress behaves comparable to other routing strategies proposed in the literature.**

I. Introduction and Model Assumptions

We investigate a packet radio network where packets have to be routed by intermediate stations to reach a target node. An important problem is the determination of transmission power (equivalently range) for each terminal in the network. Several transmission strategies have been proposed in the literature, a comprehensive presentation may be found in [4]. Assuming that stations are distributed in the plane according to a homogeneous two-dimensional Poisson process Hou and Li [4] gave an analysis of three competing strategies. In this paper and related ones [3], [5], [6], [9], [11] mathematically untractable problems were by-passed by simulation or simplifying approximations. We instead assume a one-dimensional random model which is more realistic, e.g. for road traffic information systems. Basically this means that stations are distributed according to a one-dimensional nonhomogeneous Poisson process with intensity function $\Lambda(t)$, $t \in \mathbb{R}$. For the homogeneous model, in [7] three transmission strategies have been completely analyzed: most forward and positively most forward with fixed range R, as well as positively most forward with variable transmission radius.

In this paper we investigate

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NFP: (nearest with forward progress) Each node will transmit to the nearest neighbor in the desired direction. Transmission power will be adjusted to be just strong enough to reach the receiving station.

The goal of NFP is to reduce collisions as much as possible, though the number of hops to reach a target node possibly may increase. NFP is best suited for applications in road traffic information systems such as cooperative driving by data exchange between neighboring vehicles. For this purpose large throughput is most important to achieve real time and reliable data flow. Moreover, we will see that NFP behaves very stable with respect to varying station densities, once an optimum transmission probability has been chosen. In contrast to these one-hop applications it turns out that NFP behaves slightly worse than other routing strategies, if typical multihop tasks are required. For such applications progress is an adequate measure of performance.

We now briefly outline the precise model assumptions, which in a one-dimensional environment are an extension of the ones given by [4]. The stations (nodes, terminals, vehicles) are distributed as a nonhomogeneous Poisson process ξ^Λ on $\mathbb{R}$ with increasing intensity function $\Lambda(x)$, $x \in \mathbb{R}$, normalized by $\Lambda(0) = 0$. We assume that $\Lambda(x)$ is differentiable with continuous rate $\lambda(x) = \Lambda'(x)$, i.e. the number of nodes in a fixed interval $(s, t]$ has a Poisson distribution with parameter $\int_s^t \lambda(u)du = \Lambda(t) - \Lambda(s)$. This scenery may be seen as a snapshot of randomly moving stations on a road from the point of a fixed station S positioned at the origin.

Channel access is organized by slotted ALOHA [10], and we assume that acknowledgement traffic is performed on a separate channel. We further assume that each station always has packets waiting to be transmitted (heavy traffic assumption). Collisions may occur if two or more stations with overlapping radius transmit in the same slot. In this case destroyed data packets are rescheduled at some future time. Traffic load is supposed to be uniform, expressed by the fact that every station transmits in a slot independently with probability p (transmit mode) and does not transmit with probability $1-p$ (receive mode), $0 \le p \le 1$. R will denote the maximum transmission range.

If a station is going to address a neighboring station on the right (left) we call it in right-(left-)target mode, otherwise in receive mode. This makes additional probabilities p_ℓ and p_r necessary, $p_\ell + p_r = p$, where p_ℓ (p_r) denotes the probability of finding a station in left-(right-) target mode. If a station in left- or right-target mode does not find a receiver within its transmission range, it remains in

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receive mode. This differs from the assumption in [4], [5], [11], where stations do not switch from transmit to receive mode in case they cannot find a receiving target station.

As is well known (cf. [8]), by the above assumptions the joint process ξ^Λ may be partitioned into independent corresponding nonhomogeneous Poisson processes $\xi^\Lambda_{\rm Re} + \xi^\Lambda_{\rm Tr} = \xi^\Lambda$, the transmitting and receiving process. $\xi^\Lambda_{\rm Re}$ has intensity rate $(1-p)\lambda(x)$, while $\xi^\Lambda_{\rm Tr}$ is splitted up as $\xi^\Lambda_\ell + \xi^\Lambda_r = \xi^\Lambda_{\rm Tr}$ with rates $p_\ell\lambda(x)$ and $p_r\lambda(x)$.

We emphasize that this model is local from the viewpoint of some fixed transmitter S at the origin. With constant $\lambda(x) = \lambda$ it reduces to the one considered in [7], and measures of performance may be considered globally over the whole axis. In the two dimensional model of [4], [5], [11] a target direction is chosen according to a uniform distribution. Obviously, in one dimension there are only two possible directions. The corresponding distribution is given by $p_\ell = p_r = p/2$.

In [11] the following measure of performance were introduced.

$S_{\rm NFP}(p_\ell, p_r, \Lambda)$ the one-hop throughput, defined as the expected number of successful transmissions per slot.

$Z_{\rm NFP}(p_\ell, p_r, \Lambda)$ the expected progress of a packet in the desired direction per slot from a terminal

$Z_{\rm NFP}(p_\ell, p_r, \Lambda)$ depends on the dimension of length (e.g. miles). To achieve a scale invariant measure we normalize by deviding by $M(p_\ell, p_r, \Lambda)$, the expected distance between S and its target station E in left- or right target-mode,

$$M(p_\ell, p_r, \Lambda) = \frac{p_r}{p}\int_0^\infty t\lambda(t)e^{-\Lambda(t)}\,dt - \frac{p_\ell}{p}\int_{-\infty}^0 t\lambda(t)e^{\Lambda(t)}\,dt, \tag{1.1}$$

provided the integrals exist. This holds if

$$\lim_{t\to\infty} t^{1+\varepsilon}e^{-\Lambda(t)} = 0 \text{ and } \lim_{t\to-\infty} |t|^{1+\varepsilon}e^{\Lambda(t)} = 0$$

for some $\varepsilon > 0$. (1.1) generalizes the normalization introduced in [11], it is adapted to the more general basic process. In the homogeneous case (1.1) reduces to $\frac{p_r}{p\lambda} + \frac{p_\ell}{p\lambda} = \frac{1}{\lambda}$ which is the average distance between neighboring stations.

$Z^*_{\rm NFP}(p_\ell, p_r, \Lambda) = M(p_\ell, p_r, \Lambda)^{-1}Z_{\rm NFP}(p_\ell, p_r, \Lambda)$ the normalized expected progress of a packet in the desired direction per slot from a terminal.

If the Poisson process is homogeneous it holds that $Z^*_{\rm NFP}(p_\ell, p_r, \Lambda) = \lambda Z_{\rm NFP}(p_\ell, p_r, \Lambda)$. Another scale invariant measure is achieved by taking R as normalizing distance unit (cf. [7]).

$V^*_{\rm NFP}(p_\ell, p_r, \Lambda) = \frac{1}{R}Z_{\rm NFP}(p_\ell, p_r, \Lambda)$, the relative expected progress with respect to the maximum progress R.

The paper is organized in the following way. The next chapter deals with the analytic description of $S_{\rm NFP}$ and $Z^*_{\rm NFP}$. In the Appendix two methods are described to solve the occuring differential equations. We have applied the second method to determine numerically performance measures in the homogeneous model. The behaviour of throughput and progress under NFP with optimum p-values in the homogeneous case and for a certain non-homogeneous example are presented graphically in Section III. Some concluding remarks are given in Section IV.

II. Mathematical Analysis

The subsequent investigations will depend on many disjoint subcases. To combine them under a few principles we introduce two general definitions which allow to treat all cases jointly after simple transformations (cf. Theorem 1). In the following Propositions we assume a general inhomogeneous Poisson process with intensity $\Lambda(t)$, $\Lambda(0) = 0$. Necessary shifts such that a tagged transmitting station S is fixed at the origin are finally applied in Theorem 1.

Consider a fixed realization $\xi^\Lambda(\omega)$ of the underlying nonhomogeneous Poisson process ξ^Λ and any station S in right target mode which addresses a station E to the right. Observe that in the following "left of" always means positions with smaller, and "right of" with larger coordinate values. For example, in the third system of Fig. 2.1 the station leftmost of $R - y$ is Q, and in the first system Q denotes the station leftmost of R.

Fix $x \geq 0$ and shift the origin to E. For any range R transmission from S to E is not interfered by stations in $(0, x/2)$ if for any station Q in $(0, x/2)$ there is a station Q' in $(Q, 2Q)$ (except the case that the first station rightmost to E is in left target mode). On the other hand, by reverting the coordinate system and shifting the origin to S, if $y \geq 0$ denotes the distance between S and E, transmission from S to E is not interfered by stations right of S in $(0, (x-y)/2)$ whenever each station Q in $(0, (x-y)/2)$ finds a target station Q' in $(Q, 2Q + y)$. We say that the interval $(0, x)$ is free of local y-interference. If position x is occupied by a station, as will happen later by conditioning, then stations in $(0, x)$ do not cause collisions, except the case $y = 0$. Then additionally the first station right of E must not be in left target mode. Fig. 2.1 visualizes these ideas.

The first definition characterizes patterns of stations without local interference.

Definition 1. *The interval $(0, x)$ is called free of local y-interference, $x, y \geq 0$, if for any station $Q \in \left(0, \frac{1}{2}(x-y)\right]$ there is a station $Q' \in (Q, 2Q + y)$.*

Obviously the empty interval $(0, x)$ is free of local y-interference, which is contained in Definition 1 as a special case.

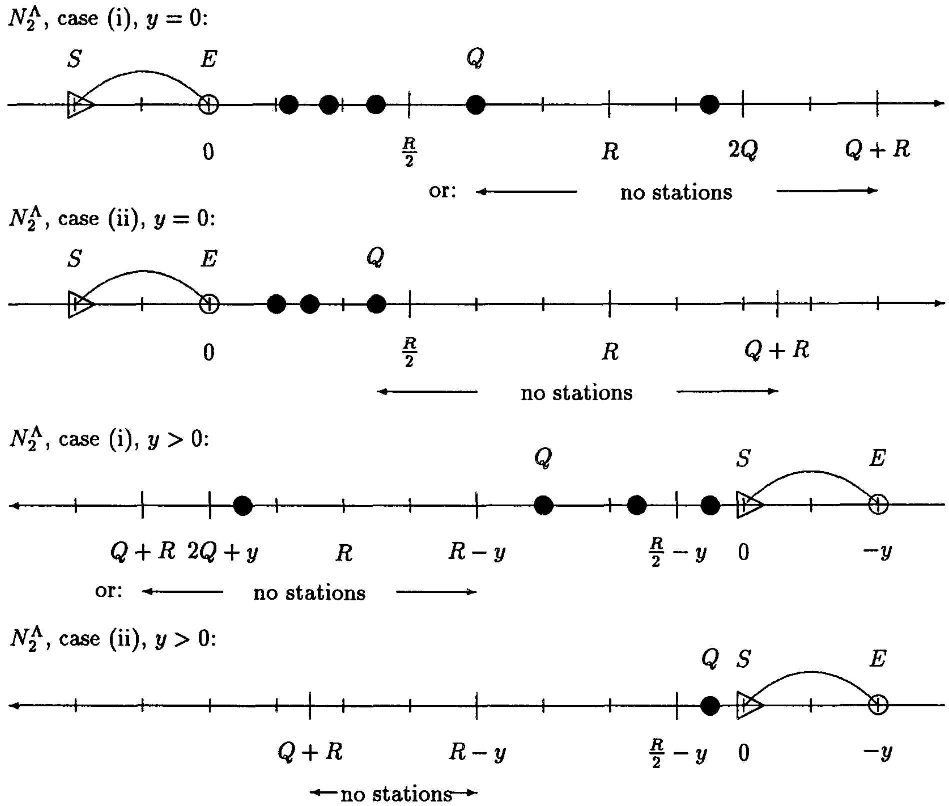

Fig. 2.1. Free of y-interference

In order to calculate recursively the probability of successful transmission we define the function $g_\Lambda(x,y) : [0,\infty) \times [0,\infty) \to [0,1]$ by

$$g_\Lambda(x,y) = P\big((0,x) \text{ is free of local } y\text{-interference}\big).$$

Proposition 1. *$g_\Lambda(x,y)$ has the following representation.*

$$g_\Lambda(x,y) = \mathbf{1}_{[0,y)}(x) + e^{-\Lambda(x)}\tilde{g}_\Lambda(x,y)\mathbf{1}_{[y,\infty)}(x), \quad x,y \geq 0, \tag{2.1}$$

where $\tilde{g}_\Lambda(x,y)$ for each $y \geq 0$ is the solution of the differential equation

$$\frac{\partial}{\partial x}\tilde{g}_\Lambda(x,y) = \lambda(x)\tilde{g}_\Lambda(x,y) - \tfrac{1}{2}\lambda\big(\tfrac{x-y}{2}\big)\tilde{g}_\Lambda\big(\tfrac{x-y}{2},y\big), \quad x \geq y \geq 0, \tag{2.2}$$

with initial condition $\tilde{g}_\Lambda(x,y) = e^{\Lambda(x)}$, $0 \leq x \leq y$.

Proof. If $y > 0$ and $0 \leq x \leq y$, by definition we have $g_\Lambda(x,y) = 1$.

Now let $0 \leq y \leq x$, and let the random variable Z_x^Λ denote the position of the first station left of x. By conditioning on $\{Z_x^\Lambda = z\}$ we obtain for all $0 \leq y \leq x$

$$\begin{aligned} g_\Lambda(x,y) &= e^{-\Lambda(x)} + \int_{\frac{x-y}{2}}^{x} P\big((0,z) \text{ free of local } y\text{-interference} \mid Z_x^\Lambda = z\big)\, dF_{Z_x^\Lambda}(z) \\ &= e^{-\Lambda(x)}\Big(1 + \int_{\frac{x-y}{2}}^{x} P\big((0,z) \text{ free of local } y\text{-interference} \mid Z_x^\Lambda = z\big)\lambda(z)e^{\Lambda(z)}dz\Big), \end{aligned}$$

where $e^{-\Lambda(x)}$ is the probability that no station is located in $(0,x)$. Using conditional independence it follows that

$$\tilde{g}_\Lambda(x,y) = 1 + \int_{\frac{x-y}{2}}^{x} \lambda(z)\tilde{g}_\Lambda(z,y)dz, \quad x \geq y \geq 0, \tag{2.3}$$

where $\tilde{g}_\Lambda(x,y) = g_\Lambda(x,y)e^{\Lambda(x)}$ is a continuous function of $x \geq 0$. Therefore we may differentiate with respect to $x \geq y$ (one sided derivative at $x = y$). This immediately leads to (2.2). The initial condition originates from the continuity of $\tilde{g}_\Lambda(\cdot,y)$. ∎

Remark. It can be shown that $g_\Lambda(x,y)$ is a continuous function of both variables in its domain $0 \leq x, y$. Furthermore, the solution $\tilde{g}_\Lambda$ of (2.2) satisfying the initial condition is unique. If $y = 0$ this follows by applying the Banach

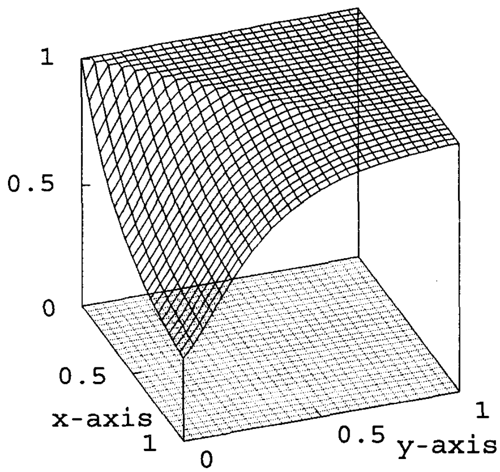

Fig. 2.2. $g_4(x,y)$

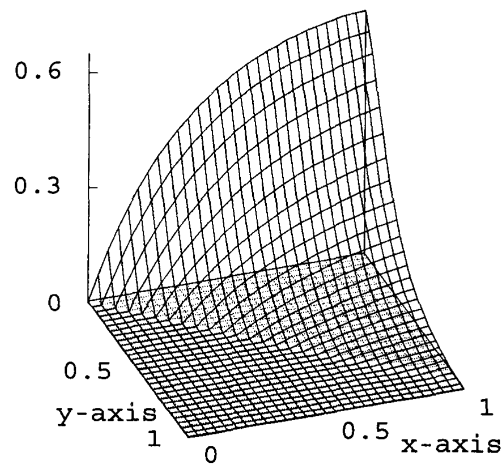

Fig. 2.3. $h_{0,0.25,4}(x,y)$

fixed-point theorem to representation (2.3), otherwise from Theorem 3 of the Appendix. Fig. 2.2 shows a plot of g_Λ in case $\Lambda(t) = 4t$.

Example. By verifying (2.1) and (2.2) it can be shown that for the intensity function $\Lambda(t) = \mathrm{sgn}(t+c)\ln\left(1+\lambda|t+c|/2\right)^2 - \ln\left(1+\lambda c/2\right)^2$, $c \geq 0, \lambda > 0,\ t \in \mathbf{R}$, the corresponding probability $g_\Lambda(x,0)$ is given by $g_\Lambda(x,0) = \left(1+\lambda c/2\right)/\left(1+\lambda(x+c)/2\right)$, $x \geq 0$. In section III we will further investigate this example.

We now describe patterns of stations which do not allow interference of transmission from S to E, provided the first station right of E is not in left-target mode. The concept "free of local interference" of Definition 1 is essentially needed. The main ideas of Definition 1 and 2 are illustrated in Fig. 2.1.

Definition 2. *The interval $(0, R-y]$, $R, y \geq 0$, is called free of y-interference if one of the following events occurs*
N_1^Λ: *$(0, R-y]$ contains no station*
or
N_2^Λ: *there is at least one station in $(0, R-y]$, the first station less than $R-y$ denoted by Q. In this case $(0,Q)$ is free of local y-interference, and*
   (i) *if $Q \geq \max\left(0, \frac{R}{2}-y\right)$ then there exists at least one station in $(R-y, 2Q+y)$ or the interval $(R-y, Q+R]$ is empty,*
   (ii) *if $0 < Q < \max\left(0, \frac{R}{2}-y\right)$ then $(R-y, Q+R)$ contains no station.*

We call the interval $(0, R-y]$ free of left y-interference if additionally in N_2^Λ the first station right of E is not in left target mode.

The relevant case of "free of left y-interference" is $y = 0$. Define the functions $G_{p_\ell,\Lambda}$, $G_\Lambda : \{(R,y) \mid 0 \leq y \leq R\} \to [0,1]$ by

$$G_{p_\ell,\Lambda}(R,y) = P\left((0,R-y] \text{ is free of left } y\text{-interference}\right)$$
and
$$G_\Lambda(R,y) = P\left((0,R-y] \text{ is free of } y\text{-interference}\right)$$

Proposition 2. *$G_{p_\ell,\Lambda}$ and G_Λ have the following explicit representations*

$$G_\Lambda(R,y) = G_{0,\Lambda}(R,y)$$
$$G_{p_\ell,\Lambda}(R,y) = e^{-\Lambda(R-y)}\Bigg(1 + \left(1 - p_\ell \mathbf{1}_{\{0\}}(y)\right) \bigg(\int_0^{R-y} \lambda(z) e^{\Lambda(z)+\Lambda(R-y)-\Lambda(R+z)} g_\Lambda(z,y)\,dz + \int_{\max(0,\frac{R}{2}-y)}^{R-y} \lambda(z) e^{\Lambda(z)} \left(1 - e^{\Lambda(R-y)-\Lambda(2z+y)}\right) g_\Lambda(z,y)\,dz\bigg)\Bigg).$$

Proof. N_1^Λ and N_2^Λ are disjoint events. Conditioning on Z_{R-y}^Λ, the position of the first station left of $R-y$, yields

$$\begin{aligned} G_{p_\ell,\Lambda}(R,y) &= P\left(N_1^\Lambda \cup N_2^\Lambda\right) \\ &= e^{-\Lambda(R-y)} + \int_0^{R-y} P\left(N_2^\Lambda \mid Z_{R-y}^\Lambda = z\right) dF_{Z_{R-y}^\Lambda}(z) \\ &= e^{-\Lambda(R-y)} + \int_0^{R-y} \lambda(z) e^{-(\Lambda(R-y)-\Lambda(z))} \\ &\quad \left(1 - p_\ell \mathbf{1}_{\{0\}}(y)\right) g_\Lambda(z,y) \Big(\left(1 - e^{-(\Lambda(2z+y)-\Lambda(R-y))}\right) \cdot \\ &\quad \mathbf{1}_{[\max(0,\frac{R}{2}-y),R-y]}(z) + e^{-(\Lambda(R+z)-\Lambda(R-y))}\Big)\,dz. \end{aligned}$$

Some further simple transformations lead to the asserted representation. G_Λ is obtained following the same lines with $p_\ell = 0$. ∎

Observe that in Fig. 2.1 for $y > 0$ stations in left-target mode to the right of S cannot interfere transmission from S to E. This means that the left target process ξ_ℓ^Λ right of S does not influence the probability of interference. Consequently, for this case $p_\ell = 0$ has to be taken in $G_{p_\ell,\Lambda}$ (see the above Proposition and its proof). The same choice, $p_\ell = 0$ whenever $y > 0$, must be used in analogous subsequent cases.

Up to now we have not taken into account target or receive mode of relevant stations. Even if the topology of stations fails to be free of y-interference it may happen that transmission from S to E is successful, due to a favorable line-up of modes. To cope with this problem we introduce two auxiliary functions of similar structure as g_Λ and $G_{p_\ell,\Lambda}$.

Let $h_{p_\ell,p_r,\Lambda} : [0,\infty) \times [0,\infty) \to [0,1]$ be defined as

$h_{p_\ell,p_r,\Lambda}(x,y) = P\big((0,x)$ has local y-interference and transmission from S to E is not interfered by stations in $(0,x)\big)$

Fig. 2.3 shows a plot of $h_{0,0.25,\Lambda}$ for $\Lambda(t) = 4t$. As in Fig. 2.1, in case $y = 0$ we fix E at 0 and S on the left of E, and in case $y > 0$ station E at $-y$ and S at 0.

Proposition 3. *$h_{p_\ell,p_r,\Lambda}(x,y)$ has the following representation.*

$$h_{p_\ell,p_r,\Lambda}(x,y) = e^{-\Lambda(x)}\tilde{h}_{p_\ell,p_r,\Lambda}(x,y)\mathbf{1}_{[y,\infty)}(x), \qquad (2.4)$$

where $\tilde{h}_{p_\ell,p_r,\Lambda}(x,y)$ for each $y \geq 0$ is the solution of the differential equation

$$\begin{aligned}\tfrac{\partial}{\partial x}\tilde{h}_{p_\ell,p_r,\Lambda}(x,y) &= \lambda(x)\tilde{h}_{p_\ell,p_r,\Lambda}(x,y) - \tfrac{1}{2}\lambda(\tfrac{x-y}{2})\\ &\Big(p_r\tilde{h}_{p_\ell,p_r,\Lambda}(\tfrac{x-y}{2},y) - (1-p_\ell\mathbf{1}_{\{0\}}(y))\\ &(1-p_r)\tilde{g}_\Lambda(\tfrac{x-y}{2},y) + p_\ell\mathbf{1}_{\{0\}}(y)p_r\Big),\ x \in \mathbf{R},\end{aligned} \qquad (2.5)$$

with initial condition $\tilde{h}_{p_\ell,p_r,\Lambda}(x,y) = 0$, $0 \leq x \leq y$.

Proof. If $y > 0$ and $0 \leq x \leq y$, then $\tilde{h}_{p_\ell,p_r,\Lambda}(x,y) = 0$ holds by definition. Let Z_x^Λ be as in the proof of Proposition 1 and Q the first station less than x. By conditioning we get for all $0 \leq y \leq x$

$h_{p_\ell,p_r,\Lambda}(x,y)$

$= \int_{\frac{x-y}{2}}^{x} P\big((0,z)$ has local y-interference and a transmission from S to E is not interfered by stations in $(0,z) \mid Z_x^\Lambda = z\big)\, dF_{Z_x^\Lambda}(z)$

$+ \int_0^{\frac{x-y}{2}} P\big($Station Q is not in right-target mode and $((0,z)$ free of local y-interference and there is a station in $(0,z))$ or $((0,z)$ has y-interference and transmission from S to E is not interfered by stations in $(0,z)) \mid Z_x^\Lambda = z\big)\, dF_{Z_x^\Lambda}(z)$

$+ \int_0^{\frac{x-y}{2}} P\big($[Station Q is in receive mode if $y = 0$ and not in right-target mode if $y > 0$] and [$(0,z)$ contains no stations] $\mid Z_x^\Lambda = z\big)\, dF_{Z_x^\Lambda}(z)$

$$\begin{aligned}&= \int_{\frac{x-y}{2}}^{x} \lambda(z)e^{-(\Lambda(x)-\Lambda(z))}h_{p_\ell,p_r,\Lambda}(z,y)\,dz\\ &+ \int_0^{\frac{x-y}{2}} \lambda(z)e^{-(\Lambda(x)-\Lambda(z))}(1-p_r)\\ &\Big(h_{p_\ell,p_r,\Lambda}(z,y) + (1-p_\ell\mathbf{1}_{\{0\}}(y))\big(g_\Lambda(z,y) - e^{-\Lambda(z)}\big)\Big)\,dz\\ &+ \int_0^{\frac{x-y}{2}} \lambda(z)e^{-(\Lambda(x)-\Lambda(z))}\big(1-p_r-p_\ell\mathbf{1}_{\{0\}}(y)\big)e^{-\Lambda(z)}\,dz.\end{aligned}$$

The second term in the sum is a bit complicated. $g_\Lambda(z,y)$ is the probability that $(0,z)$ is free of local y-interference, which contains the event that $(0,z)$ is free of stations. The corresponding probability has to be subtracted from $g_\Lambda(z,y)$. If $y = 0$ the first station rightmost of E must not be in left transmit mode which explains the factor $(1 - p_\ell\mathbf{1}_{\{0\}}(y))$.

With $\tilde{h}_{p_\ell,p_r,\Lambda}(x,y) = e^{\Lambda(x)}h_{p_\ell,p_r,\Lambda}(x,y)$ we obtain

$$\begin{aligned}\tilde{h}_{p_\ell,p_r,\Lambda}(x,y) &= \int_{\frac{x-y}{2}}^{x} \lambda(z)\tilde{h}_{p_\ell,p_r,\Lambda}(z,y)\,dz + (1-p_r)\\ &\int_0^{\frac{x-y}{2}} \lambda(z)\Big(\tilde{h}_{p_\ell,p_r,\Lambda}(z,y) + (1-p_\ell\mathbf{1}_{\{0\}}(y))\tilde{g}_\Lambda(z,y)\Big)\,dz\\ &- p_\ell\mathbf{1}_{\{0\}}(y)p_r\Lambda(\tfrac{x-y}{2}).\end{aligned}$$

Since $\tilde{h}_{p_\ell,p_r,\Lambda}(x,y)$ is a continuous function of $x \geq 0$ we may differentiate with respect to $x \geq y$ (one sided derivative at $x = y$). This leads to the differential equation (2.5). The initial condition originates from the continuity of $\tilde{h}_{p_\ell,p_r,\Lambda}(\cdot,y)$. ∎

We now consider the probability that transmission from S to E is possible, in spite of an unfavourable line-up of stations. Define the function $H_{p_\ell,p_r,\Lambda}(R,y) : \{(R,y) \mid 0 \leq y \leq R\} \to [0,1]$ by

$H_{p_\ell,p_r,\Lambda}(R,y) = P\big((0,R-y]$ has left y-interference and transmission from S to E is not interfered by stations in $(0,R-y]\big)$.

Proposition 4. *It holds that*

$$\begin{aligned}&H_{p_\ell,p_r,\Lambda}(R,y)\\ &= e^{-\Lambda(R-y)}\bigg\{\int_{\min(y,R-y)}^{R-y} \lambda(z)e^{\Lambda(z)+\Lambda(R-y)-\Lambda(R+z)}\\ &h_{p_\ell,p_r,\Lambda}(z,y)\,dz + \int_{\max(\frac{R}{2}-y,0)}^{R-y} \lambda(z)e^{\Lambda z}\\ &\big(1-e^{\Lambda(R-y)-\Lambda(2z+y)}\big)h_{p_\ell,p_r,\Lambda}(z,y)\,dz\end{aligned}$$

$$
\begin{aligned}
&+(1-p_r)\int_{\max(\frac{R}{2}-y,0)}^{R-y}\lambda(z)e^{\Lambda(z)+\Lambda(R-y)}\\
&\quad\left(e^{-\Lambda(2z+y)}-e^{-\Lambda(R+z)}\right)\left(h_{p_\ell,p_r,\Lambda}(z,y)\right.\\
&\quad\left.+\left(1-p_\ell\mathbf{1}_{\{0\}}(y)\right)g_\Lambda(z,y)\right)dz\\
&+(1-p_r)\int_0^{\max(\frac{R}{2}-y,0)}\lambda(z)e^{\Lambda(z)}\\
&\quad\left(1-e^{\Lambda(R-y)-\Lambda(R+z)}\right)\left(h_{p_\ell,p_r,\Lambda}(z,y)\right.\\
&\quad\left.+\left(1-p_\ell\mathbf{1}_{\{0\}}(y)\right)g_\Lambda(z,y)\right)dz\\
&-p_\ell\mathbf{1}_{\{0\}}(y)p_r\left(\int_0^{\max(\frac{R}{2}-y,0)}\lambda(z)\left(1-e^{\Lambda(R-y)-\Lambda(R+z)}\right)dz\right.\\
&\quad\left.+\int_{\max(\frac{R}{2}-y,0)}^{R-y}\lambda(z)e^{\Lambda(R-y)}\left(e^{-\Lambda(2z+y)}-e^{-\Lambda(R+z)}\right)dz\right)\Bigg\}.
\end{aligned}
$$

Proof. Let $Q\in(0,R-y]$ be the first station left of $R-y$ such that $Z^\Lambda_{R-y}=Q$. We distinguish two disjoint cases:

1) Q is unable to produce interference, with two subcases:

$$S^\Lambda_{11}=\{(R-y,R+Q]\text{ is free of stations}\},$$
$$S^\Lambda_{12}=\{\text{there is a station in }(R-y,y+2Q]\}.$$

2) Q is able to produce interference, again with two subcases:

$$S^\Lambda_{21}=\{\max(\tfrac{R}{2}-y,0)\le Q,\text{ no station in }(R-y,y+2Q),\text{ and at least one station in }[y+2Q,R+Q]\},$$
$$S^\Lambda_{22}=\{0<Q\le\max(\tfrac{R}{2}-y,0)\text{ and there is a station in }(R-y,R+Q]\}.$$

By summing the probabilities of the following disjoint events we get

$$
\begin{aligned}
&H_{p_\ell,p_r,\Lambda}(R,y)\\
&=P\big((S^\Lambda_{11}\cup S^\Lambda_{12}\cup S^\Lambda_{21}\cup S^\Lambda_{22})\cap\{(0,R-y]\text{ has left }y\text{-interference and transmission from }S\text{ to }E\text{ is possible }\}\big)\\
&=\int_0^{R-y}\lambda(z)e^{-(\Lambda(R-y)-\Lambda(z))}e^{-(\Lambda(R+z)-\Lambda(R-y))}\\
&\quad h_{p_\ell,p_r,\Lambda}(z,y)\,dz+\int_{\max(\frac{R}{2}-y,0)}^{R-y}\lambda(z)e^{-(\Lambda(R-y)-\Lambda(z))}\\
&\quad\left(1-e^{-(\Lambda(2z+y)-\Lambda(R-y))}\right)h_{p_\ell,p_r,\Lambda}(z,y)\,dz\\
&+(1-p_r)\int_{\max(\frac{R}{2}-y,0)}^{R-y}\lambda(z)e^{-(\Lambda(R-y)-\Lambda(z))}\\
&\quad e^{-(\Lambda(2z+y)-\Lambda(R-y))}\left(1-e^{-(\Lambda(R+z)-\Lambda(2z+y))}\right)\\
&\quad\left(h_{p_\ell,p_r,\Lambda}(z,y)+\left(1-p_\ell\mathbf{1}_{\{0\}}(y)\right)\left(g_\Lambda(z,y)-e^{-\Lambda(z)}\right)\right)dz\\
&+\left(1-p_r-p_\ell\mathbf{1}_{\{0\}}(y)\right)\int_{\max(\frac{R}{2}-y,0)}^{R-y}\lambda(z)e^{-(\Lambda(R-y)-\Lambda(z))}\\
&\quad e^{-(\Lambda(2z+y)-\Lambda(R-y))}\left(1-e^{-(\Lambda(R+z)-\Lambda(2z+y))}\right)e^{-\Lambda(z)}\,dz\\
&+(1-p_r)\int_0^{\max(\frac{R}{2}-y,0)}\lambda(z)e^{-(\Lambda(R-y)-\Lambda(z))}\\
&\quad\left(1-e^{-(\Lambda(R+z)-\Lambda(R-y))}\right)\\
&\quad\left(h_{p_\ell,p_r,\Lambda}(z,y)+\left(1-p_\ell\mathbf{1}_{\{0\}}(y)\right)\left(g_\Lambda(z,y)-e^{-\Lambda(z)}\right)\right)dz\\
&+\left(1-p_r-p_\ell\mathbf{1}_{\{0\}}(y)\right)\int_0^{\max(\frac{R}{2}-y,0)}\lambda(z)e^{-(\Lambda(R-y)-\Lambda(z))}\\
&\quad\left(1-e^{-(\Lambda(R+z)-\Lambda(R-y))}\right)e^{-\Lambda(z)}\,dz.
\end{aligned}
$$

We have six additive terms. The first one corresponds to S^Λ_{11}, the second one to S^Λ_{12}, the first pair of the remaining ones to S^Λ_{21}, and the last pair to S^Λ_{22}, each intersected with the event {transmission from S to E is possible}. In deriving this formula the individual cases have to be considered very carefully. Observe that $h_{p_\ell,p_r,\Lambda}(z,y)=0$ if $0\le z\le y$. Some further algebra yields the assertion. ■

We are now prepared to prove the main result of this section.

Theorem 1. *Let Λ, $p_\ell+p_r=p$ be the parameters of the underlying nonhomogeneous Poisson process ξ^Λ, and R denote the transmission range. The one-hop throughput $S_{\mathrm{NFP}}(p_\ell,p_r,\Lambda)$ and the normalized expected progress $Z^*_{\mathrm{NFP}}(p_\ell,p_r,\Lambda)$ satisfy*

$$
\begin{aligned}
&S_{\mathrm{NFP}}(p_\ell,p_r,\Lambda)\\
&=p_r\int_0^R\lambda(y)e^{-\Lambda(y)}\left(G_{-\Lambda(-\cdot)}(R,y)+H_{0,p_\ell,-\Lambda(-\cdot)}(R,y)\right)\\
&\quad\Big\{(1-p)\big(G_{p_\ell,\Lambda(\cdot+y)-\Lambda(y)}(R,0)\\
&\quad+H_{p_\ell,p_r,\Lambda(\cdot+y)-\Lambda(y)}(R,0)\big)+p_re^{-(\Lambda(R+y)-\Lambda(y))}\Big\}\,dy\\
&+p_\ell\int_{-R}^0\lambda(y)e^{\Lambda(y)}\left(G_\Lambda(R,-y)+H_{0,p_r,\Lambda}(R,-y)\right)\\
&\quad\Big\{(1-p)\big((G_{p_r,-(\Lambda(-\cdot+y)-\Lambda(y))}(R,0)\\
&\quad+H_{p_r,p_\ell,-(\Lambda(-\cdot+y)-\Lambda(y))}(R,0)\big)\\
&\quad+p_\ell e^{-(\Lambda(y)-\Lambda(-R+y))}\Big\}\,dy
\end{aligned}
$$

$$
\begin{aligned}
&Z^*_{\mathrm{NFP}}(p_\ell,p_r,\Lambda)\\
&=\left(M(p_\ell,p_r,\Lambda)\right)^{-1}p_r\int_0^R y\lambda(y)e^{-\Lambda(y)}\\
&\quad\left(G_{-\Lambda(-\cdot)}(R,y)+H_{0,p_\ell,-\Lambda(-\cdot)}(R,y)\right)\Big\{(1-p)\\
&\quad\left(G_{p_\ell,\Lambda(\cdot+y)-\Lambda(y)}(R,0)+H_{p_\ell,p_r,\Lambda(\cdot+y)-\Lambda(y)}(R,0)\right)\\
&\quad+p_re^{-(\Lambda(R+y)-\Lambda(y))}\Big\}\,dy\\
&+p_\ell\int_{-R}^0(-y)\lambda(y)e^{\Lambda(y)}\left(G_\Lambda(R,-y)+H_{0,p_r,\Lambda}(R,-y)\right)\\
&\quad\Big\{(1-p)\big((G_{p_r,-(\Lambda(-\cdot+y)-\Lambda(y))}(R,0)\\
&\quad+H_{p_r,p_\ell,-(\Lambda(-\cdot+y)-\Lambda(y))}(R,0)\big)\\
&\quad+p_\ell e^{-(\Lambda(y)-\Lambda(-R+y))}\Big\}\,dy
\end{aligned}
$$

with $M(p_\ell, p_r, \Lambda)$ *from (1.1).*

Proof. The basic idea in the following proof is to shift and mirror the intensity function Λ. Since S is fixed at the origin this allows to apply Propositions 2 and 4 easily.

Define the event $M_r^\Lambda = \{S$ in right-target mode, transmission of S is sucessful$\}$. With S in right target mode, S fixed at the origin, let the random variable $X_{\mathrm{NFP}}^{S,r,\Lambda}$ denote the position of the first station on the right of S (the position of E in case $0 < y \le R$). Conditioning on $X_{\mathrm{NFP}}^{S,r,\Lambda}$ and splitting up into disjoint events yields

$$\begin{aligned}
&P(M_r^\Lambda \mid X_{\mathrm{NFP}}^{S,r,\Lambda}) \\
&= P(S \text{ in right-target mode, } E \text{ in receive mode, transmission from } S \text{ to } E \text{ is not interfered by stations on the left of } S \text{ and on the right of } E) \\
&\quad + P(S \text{ and } E \text{ in right-target mode, no station in } [y, y+R], \text{ and transmission from } S \text{ to } E \text{ is not interfered by stations on the left of } S) \\
&= p_r(1-p)\Big(G_{p_\ell,\Lambda(\cdot+y)-\Lambda(y)}(R,0) \\
&\qquad + H_{p_\ell,p_r,\Lambda(\cdot+y)-\Lambda(y)}(R,0)\Big) \\
&\qquad \Big(G_{-\Lambda(-\cdot)}(R,y) + H_{0,p_\ell,-\Lambda(-\cdot)}(R,y)\Big) \\
&\qquad + p_r^2 e^{-(\Lambda(R+y)-\Lambda(y))}\big(G_{-\Lambda(-\cdot)}(R,y) \\
&\qquad + H_{0,p_\ell,-\Lambda(-\cdot)}(R,y)\big).
\end{aligned}$$

Defining the event M_ℓ^Λ and the random variable $X_{\mathrm{NFP}}^{S,\ell,\Lambda}$ analogously, the final result follows from

$$\begin{aligned}
S_{\mathrm{NFP}}(p_\ell, p_r, \Lambda) &= P(M_r^\Lambda) + P(M_\ell^\Lambda) \\
&= \int_0^R P(M_r^\Lambda \mid X_{\mathrm{NFP}}^{S,r,\Lambda} = y)\, dF_{X_{\mathrm{NFP}}^{S,r,\Lambda}}(y) \\
&\quad + \int_{-R}^0 P(M_\ell^\Lambda \mid X_{\mathrm{NFP}}^{S,\ell,\Lambda} = y)\, dF_{X_{\mathrm{NFP}}^{S,\ell,\Lambda}}(y).
\end{aligned}$$

By multiplying each integrand of S_{NFP} by $M(p_\ell, p_r, \Lambda)^{-1}$ and $|y|$, after some algebra we obtain the corresponding expression for Z_{NFP}^*. ∎

With Theorem 1 we have arrived at rather complicated terms for throughput and normalized expected progress. To evaluate these expressions numerically the differential equations in Proposition 1 and 3 have to be solved. This is carried out in the Appendix for the special case $\Lambda(t) = \lambda t$, $\lambda > 0$ a constant, $t \ge 0$. Furthermore, we have prepared subroutines to calculate the probabilities in Proposition 2, Proposition 4, and Theorem 1 for varying parameters p_ℓ and p_r. In section III these are applied to obtain numerical results.

The remaining part of this section deals with the homogeneous case $\Lambda(t) = \lambda t$, $t \in \mathbf{R}, \lambda > 0$. Significant simplifications turn out. We first consider $p_r = p_\ell = \frac{p}{2}$, $p \in [0,1]$, which yields an analogous model to e.g. [4] in the one-dimensional case. The following analytic representation of throughput and normalized expected progress is deduced from Theorem 1, for notational convenience we use subscript λ instead of $\lambda\cdot$.

Corollary 1. *Let* $p_r = p_\ell = \frac{p}{2}$, $p \in [0,1]$, $\lambda(t) = \lambda > 0$, $t \in \mathbf{R}$, $R > 0$, *and* $N = \lambda R$. *Then* $M(p_\ell, p_r, \Lambda) = \lambda$ *in (1.1) and*

$$\begin{aligned}
&S_{\mathrm{NFP}}(\tfrac{p}{2}, \tfrac{p}{2}, \lambda) \\
&= p \int_0^R \lambda e^{-\lambda y}\Big(G_\lambda(R,y) + H_{0,p/2,\lambda}(R,y)\Big)\, dy \\
&\qquad \Big((1-p)\big(G_{p/2,\lambda}(R,0) + H_{p/2,p/2,\lambda}(R,0)\big) + \tfrac{p}{2} e^{-N}\Big) \\
&Z_{\mathrm{NFP}}^*(\tfrac{p}{2}, \tfrac{p}{2}, \lambda) \\
&= p \int_0^R \lambda^2\, y\, e^{-\lambda y}\Big(G_\lambda(R,y) + H_{0,p/2,\lambda}(R,y)\Big)\, dy \\
&\qquad \Big((1-p)\big(G_{p/2,\lambda}(R,0) + H_{p/2,p/2,\lambda}(R,0)\big) + \tfrac{p}{2} e^{-N}\Big).
\end{aligned}$$

Numerical evaluations of the formulae in Theorem 1 for the homogeneous model show that one sided transmission ($p_\ell = 0$, $p_r = p$) maximizes throughput and normalized expected progress. The assertions of Corollary 2 are derived from Theorem 1 taking into account that $G_\lambda(x,y) + H_{0,0,\lambda}(x,y) = 1$ (cf. the Appendix).

Corollary 2. *Let* $p_\ell = 0$, $p_r = p$. *Then*

$$\begin{aligned}
S_{\mathrm{NFP}}(0,p,\lambda) &= p\,(1 - e^{-N}) \\
&\qquad \Big((1-p)\big(G_\lambda(R,0) + H_{0,p,\lambda}(R,0)\big) + p e^{-N}\Big) \\
Z_{\mathrm{NFP}}^*(0,p,\lambda) &= p\,(1 - e^{-N} - N e^{-N}) \\
&\qquad \Big((1-p)\big(G_\lambda(R,0) + H_{0,p,\lambda}(R,0)\big) + p e^{-N}\Big).
\end{aligned}$$

From Theorem 2 of the Appendix it is easily seen that $S_{\mathrm{NFP}}(0,p,\lambda)$ and $Z_{\mathrm{NFP}}^*(0,p,\lambda)$ depend on λ and R only through $N = \lambda R$. Both terms are convergent when N tends to infinity. In contrast to the results in [7] both limits coincide. To determine the limiting value we define the functions $m_n(p)$ and $m(p)$, $0 \le p \le 1$, by

$$\begin{aligned}
m_n(p) &= \prod_{i=1}^n \Big(1 - \frac{p}{2^i}\Big), \quad n \in \mathbf{N}_0, \quad \text{and} \\
m(p) &= \prod_{i=1}^\infty \Big(1 - \frac{p}{2^i}\Big).
\end{aligned} \tag{2.6}$$

Using the solutions of the differential equations (2.2) and (2.5) in the case $y = 0$ (cf. the Appendix) we obtain the following representation of this limit.

Corollary 3. *For increasing network connectivity* $N = \lambda R$, *for any* $p \in [0,1]$ *it holds that*

$$\begin{aligned}
&\lim_{N\to\infty} S_{\mathrm{NFP}}(0,p,\lambda) = \lim_{N\to\infty} Z_{\mathrm{NFP}}^*(0,p,\lambda) \\
&\quad = p\,(1-p)\Big(m(1) + (1-p)\, m(p) \sum_{i=1}^\infty \frac{m_{i-1}(1)}{2^i\, m_i(p)}\Big).
\end{aligned}$$

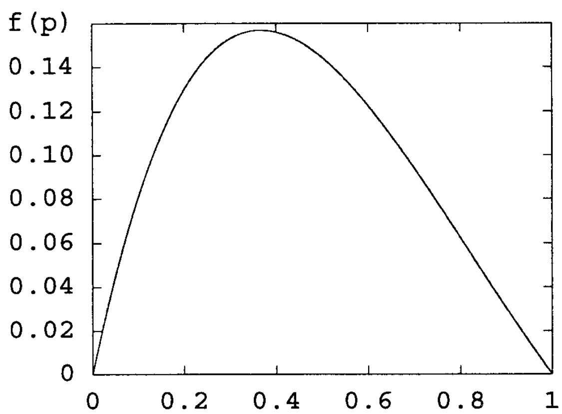

Fig. 2.4. $f(p)$

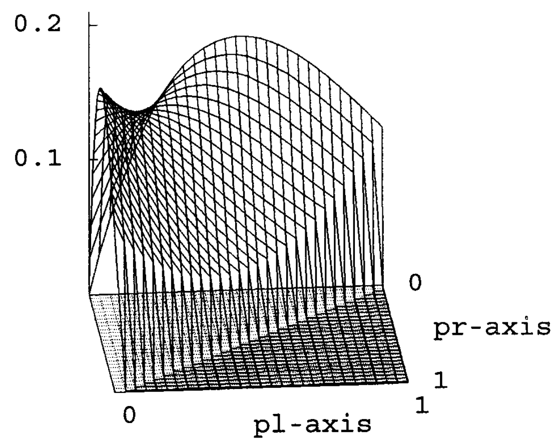

Fig. 3.1. $S_{\mathrm{NFP}}(p_\ell, p_r, 2)$

Consider the right hand side of Corollary 3 as a function f of p, $0 \le p \le 1$. This function is depicted in Fig. 2.4. The maximum over p is attained at $p^* = 0.367$ with value 0.157. For MFR-like routing strategies RS the corresponding terms $\lim_{N\to\infty} \sup_{0\le p\le 1} S_{\mathrm{RS}}(0,p,\lambda)$ are zero, while $\lim_{N\to\infty} \sup_{0\le p\le 1} Z^*_{\mathrm{RS}}(0,p,\lambda) = 1/2e = 0.184 > 0.157$ (cf. [7]).

This shows that for high station densities the MFR-type limits of normalized expected progress are superior to the corresponding NFP-limit.

III. Numerical Results

For the homogeneous case $(\Lambda(t) = \lambda t)$ we now present the results of the preceding section graphically. The corresponding curves will give a thorough insight into the behaviour of NFP. Some remarks concerning applications in road traffic information systems will also be given.

First we have examined numerically the behaviour of $S_{\mathrm{NFP}}(p_\ell, p_r, \lambda)$ as a function of p_ℓ and p_r in the domain $0 \le p_\ell, p_r, p = p_\ell + p_r \le 1$, with λ fixed. The corresponding graph for $\lambda = 2$ is depicted in Fig. 3.1. The function surface is saddle shaped. On diagonal lines $p_\ell + p_r = p =$ const we see convex behaviour with a unique minimum at $p_\ell = p_r = p/2$. This shows that throughput is largest if $p_\ell = 0$ and $p_r = p$ (or vice versa), and smallest if $p_\ell = p_r = p/2$ for any $p \in [0,1]$.

The same behaviour turns out for any value $\lambda > 0$, and as well for normalized and expected progress. Thus, the $(0,p)$- and $(\frac{p}{2}, \frac{p}{2})$-model are the most outstanding cases which are investigated further. It can be shown that the considered performance measures depend on λ and R only through $N = \lambda R$.

Fig. 3.2 deals with the $(0,p)$-model (best case). The solid, dotted, and dashed line show for optimum p-values

$$\sup_{0\le p\le 1} S(0,p,2N), \quad \sup_{0\le p\le 1} Z^*(0,p,2N), \quad \text{and} \sup_{0\le p\le 1} V^*(0,p,2N),$$

respectively. The x-axis is scaled in units $2N$ to make the results comparable with [4], [7], and [11]. It can be seen that for throughput and relative expected progress magic numbers exist, namely maximum throughput 0.25 at $2N = 1.4$ and maximum relative progress 0.11 at $2N = 1.2$ with $p = 1$ in both cases. The corresponding optimum p_r-values are shown by the solid line in Fig. 3.4. From Corollary 2 it is easily seen that these values coincide for throughput and normalized progress, and obviously for relative expected progress as well.

Remember from Corollary 3 that the limits of both throughput and normalized progress are 0.157. The curves show that this limit is attained very rapidly.

Along the same lines we have treated the $(\frac{p}{2}, \frac{p}{2})$-model in Fig. 3.3. Magic numbers occur at $2N = 2.9$, $p_r = 0.235$ with $S = 0.15$, and at $2N = 1.8$, $p_r = 0.297$ with $V^* = 0.064$. In contrast to the above, the corresponding optimum p-values slightly differ for throughput and progress (see Fig. 3.4).

Some significant differences to the results of [4] turn out. For the two-dimensional model of Hou and Li normalized expected progress is uniformly larger compared with MFR^+ and MVR^+, in contrast to our results which show MFR-like routing strategies superior. This could be due to dimensionality differences or rough interpolation of simulation results in [4]. Moreover, the asymptotically optimum p in [4], Fig. 10 is 0.32, approximately 0.05 smaller than ours.

In the next step we have fixed the asymptotically optimum $p^* = 0.367$ (cf. Corollary 3) for the $(0,p)$-model. Throughput $S(0,p^*,2N)$ and progress $Z^*(0,p^*,2N)$ are of course smaller than with optimum p-values. This is clearly

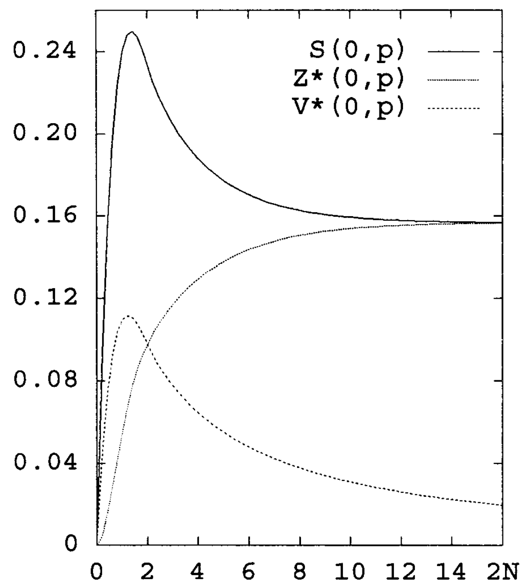

Fig. 3.2. Performance measures, $(0, p)$-model

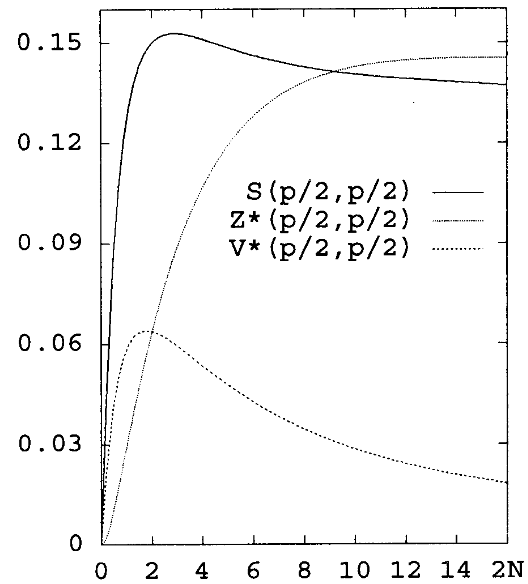

Fig. 3.3. Performance measures, $(\frac{p}{2}, \frac{p}{2})$-model

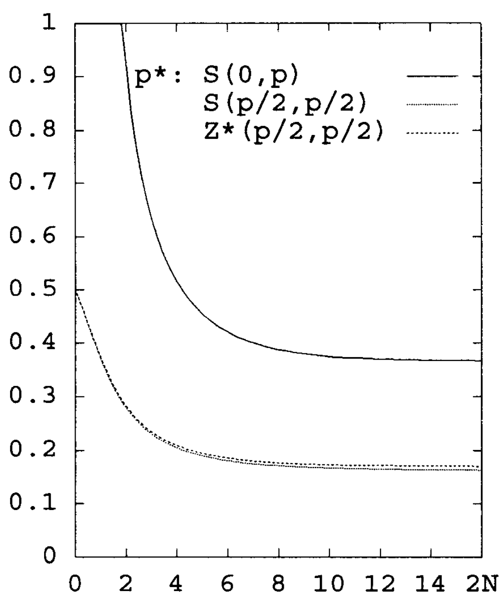

Fig. 3.4. Optimum p_r-values

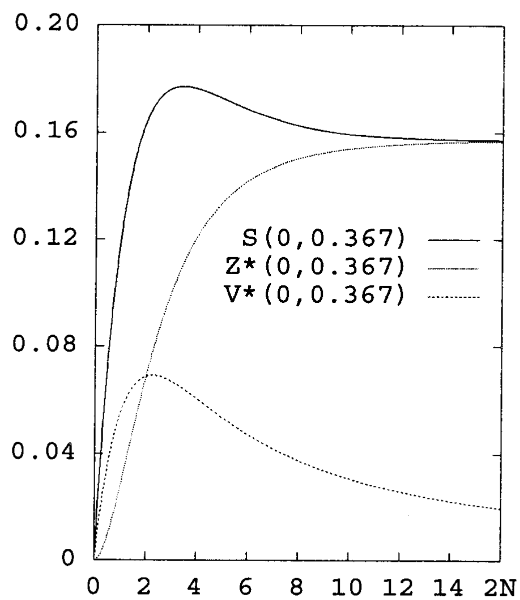

Fig. 3.5. Performance measures with asympt. opt. p^*

seen from Fig. 3.5. Surprisingly the curves differ not much from the corresponding ones in Fig. 3.2. This proves strong robustness of NFP in the $(0, 0.367)$-model against varying station densities. The value $p^* = 0.367$ is a uniformly well behaving transmission probability. With that p^*, nearly each second transmitted packet is successful for any station density $N \geq 2$.

In order to realize the advantage of the $(0, p)$-model in cooperative driving, available slots should be devided into two groups, odd and even ones, say. All stations should use slots of the same group to transmit into the same direction, alternatively to their right and left neighbor stations. Our analysis recommends constant transmission probability $p^* = 0.367$ to achieve stable system behaviour against varying station densities.

We have also investigated an example of a non-homogeneous underlying Poisson process with intensity function $\Lambda(t) = \mathrm{sgn}(t) \ln \left(1 + |t|/2\right)^2$ and $M(p_\ell, p_r, \Lambda) = 2$. This

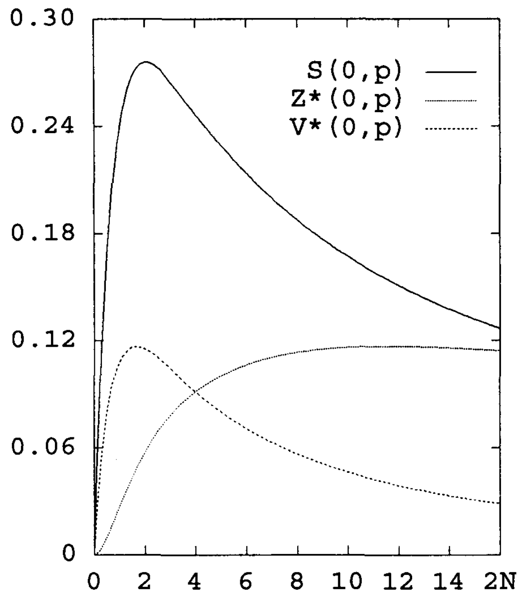

Fig. 3.6. Performance measures, $(0,p)$-model, non-homogeneous case

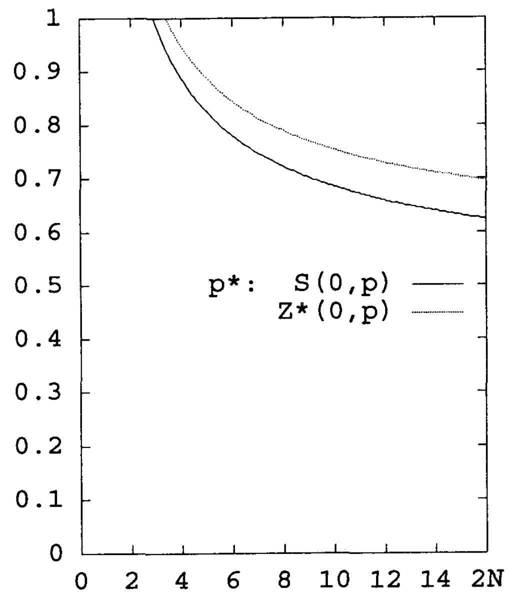

Fig. 3.7. Optimum p-values, $(0,p)$-model, non-homogeneous case

means decreasing station density with growing distance from the reference station. As in the homogeneous model it turns out that $p_\ell = 0$ and $p_r = p$ (and vice versa) maximize performance. For numerical purposes we can use the representation of $g_\Lambda(x,0)$ in Example 1. The corresponding results are shown in Fig. 3.6 and 3.7. Qualitatively we observe a similar behaviour as in the homogeneous case. The limits of the optimum $S(0,p)$ and $Z^*(0,p)$ (with $N \to \infty$) have changed to approximately 0.12, smaller than the corresponding 0.157 in Fig. 3.2. Magic numbers occur at $2N = 2.1$, $p = 1$ with $S = 0.276$, and $2N = 1.7$, $p = 1$ with $V^* = 0.116$.

IV. Conclusions

We have presented an analysis of routing strategy NFP in a generalized one dimensional model which is of most interest in applications. On the basis of this model a full analytic description of important performance measures has been given. We have found nice stability properties of NFP concerning variations of traffic density, in contrast to MFR-like strategies. There are some differences to the analysis given for the usual two dimensional model, but one should visualize that an exact analysis in two dimensions is still an open problem.

In this paper we have thoroughly investigated homogeneous and nonhomogeneous traffic. Suitable intensity functions Λ may be obtained from fitting inhomogeneous Poisson processes to statistical data. Future work will be devoted to this task.

Appendix

We now solve the differential equations (2.2) and (2.5) if the underlying Poisson process ξ^Λ is homogeneous with $\Lambda(t) = \lambda t$, $t \in \mathbb{R}$, $\lambda > 0$. The cases $y = 0$ and $y > 0$ turn out to have essential structural differences. So we will treat them separately.

Theorem 2. *Let $y = 0$ and $\Lambda(t) = \lambda t$, $t \in \mathbb{R}$, $\lambda > 0$. Then*

a)
$$\tilde{g}_\lambda(x,0) = \sum_{k=0}^{\infty} m_k(1) \frac{(\lambda x)^k}{k!}, \tag{4.1}$$

with $m_k(p)$ from (2.6) solves the differential equation (2.2).

b) *The solution of the differential equation (2.5) is given by*

$$\tilde{h}_{p_\ell,p_r,\lambda}(x,0) = \sum_{k=1}^{\infty} b_k(p_l,p_r) \frac{(\lambda x)^k}{k!}, \tag{4.2}$$

where

$$b_k(p_\ell,p_r) = \Big((1-p_\ell)(1-p_r) \sum_{i=1}^{k} \frac{m_{i-1}(1)}{2^i\, m_i(p_r)} - \frac{p_\ell p_r}{2\, m_1(p_r)}\Big) m_k(p_r).$$

Proof. a) $\tilde{g}_\lambda(x,0)$ fulfills the initial condition $\tilde{g}_\lambda(0,0)=1$

and it holds that

$$\begin{aligned}\tilde{g}'_\lambda(x,0) &= \lambda \sum_{k=0}^{\infty} m_{k+1}(1)\frac{(\lambda x)^k}{k!}\\ &= \lambda \sum_{k=0}^{\infty}\Big(m_k(1) - \frac{m_k(1)}{2^{k+1}}\Big)\frac{(\lambda x)^k}{k!}\\ &= \lambda\big(\tilde{g}_\lambda(x,0) - \tfrac{1}{2}\tilde{g}_\lambda(\tfrac{x}{2},0)\big).\end{aligned}$$

b) Clearly $\tilde{h}_{p_\ell,p_r,\lambda}(0,0) = 0$ holds, and furthermore for the derivative $\tilde{h}'_{p_\ell,p_r,\lambda}(x,0) = \lambda\sum_{k=0}^{\infty} b_{k+1}(p_\ell,p_r)\frac{(\lambda x)^k}{k!}$. By comparing the coefficients of the corresponding power series in (2.5) we obtain for $k=0$

$$\begin{aligned}b_1(p_\ell,p_r) &= \frac{(1-p_\ell)(1-p_r)}{2} - \frac{p_\ell p_r}{2}\\ &= \frac{(1-p_\ell)(1-p_r)m_0(1)}{2} - \frac{p_\ell p_r}{2}\end{aligned}$$

and for $k>0$

$$\begin{aligned}&b_{k+1}(p_\ell,p_r)\\ &= \Big((1-p_\ell)(1-p_r)\sum_{i=1}^{k+1}\frac{m_{i-1}(1)}{2^i\, m_i(p_r)} - \frac{p_\ell p_r}{2\, m_1(p_r)}\Big)m_{k+1}(p_r)\\ &= \Big((1-p_\ell)(1-p_r)\sum_{i=1}^{k}\frac{m_{i-1}(1)}{2^i\, m_i(p_r)} - \frac{p_\ell p_r}{2\, m_1(p_r)}\Big)m_k(p_r)\\ &\quad - \frac{p_r}{2}\Big((1-p_\ell)(1-p_r)\sum_{i=1}^{k}\frac{m_{i-1}(1)}{2^i\, m_i(p_2)} - \frac{p_\ell p_r}{2\, m_1(p_r)}\Big)\\ &\quad \frac{m_k(p_r)}{2^k} + \frac{(1-p_\ell)(1-p_r)}{2}\,\frac{m_k(1)}{2^k}\\ &= b_k(p_\ell,p_r) - \frac{p_r}{2^{k+1}}b_k(p_\ell,p_r) + \frac{(1-p_\ell)(1-p_r)}{2^{k+1}}m_k(1).\end{aligned}$$

Using the above identity, differentiation yields

$$\begin{aligned}&\tfrac{\partial}{\partial x}\tilde{h}_{p_\ell,p_r,\lambda}(x,0)\\ &= \lambda\sum_{k=0}^{\infty} b_{k+1}(p_\ell,p_r)\frac{(\lambda x)^k}{k!}\\ &= \lambda\Big(\sum_{k=1}^{\infty} b_k(p_\ell,p_r)\frac{(\lambda x)^k}{k!} - \frac{p_r}{2}\sum_{k=1}^{\infty}\frac{b_k(p_\ell,p_r)}{2^k}\frac{(\lambda x)^k}{k!}\\ &\quad + \frac{(1-p_\ell)(1-p_r)}{2}\sum_{k=0}^{\infty}\frac{m_k(1)}{2^k}\frac{(\lambda x)^k}{k!} - \frac{p_\ell p_r}{2}\Big)\\ &= \lambda\Big(\tilde{h}_{p_\ell,p_r,\lambda}(x,0) - \frac{p_r}{2}\tilde{h}_{p_\ell,p_r,\lambda}(\tfrac{x}{2},0)\\ &\quad + \frac{(1-p_\ell)(1-p_r)}{2}\tilde{g}_\lambda(\tfrac{x}{2},0) - \frac{p_\ell p_r}{2}\Big).\end{aligned}$$

∎

Theorem 2 allows to validitate the correctness of our considerations. From Definition 1 and 2 it follows that

$$g_\lambda(x,0) + h_{0,0,\lambda}(x,0) = 1 \quad\text{and}$$
$$G_\lambda(R,0) + H_{0,0,\lambda}(R,0) = 1.$$

The first identity holds since $m_k(1) + \sum_{i=1}^{k}\frac{m_{i-1}(1)}{2^i} = 1$, as may be shown by induction. Elementary calculations show that $G_\lambda(R,0)$ and $H_{0,0,\lambda}(R,0)$ in Proposition 2 and 4 in fact sum up to 1.

Theorem 3. *Let $y>0$ and $\Lambda(t) = \lambda t$, $t\in\mathbf{R}$, $\lambda>0$.*

a)

$$g_\lambda(x,y) = e^{-\lambda x}\sum_{n=0}^{\infty}\tilde{g}_{\lambda,n}(x,y)\mathbf{1}_{[(2^n-1)y,2^{n+1}-1)y)}(x), \quad (4.3)$$

where $\tilde{g}_{\lambda,0}(x,y) = e^{\lambda x}$, $x\in\mathbf{R}$, and

$$\begin{aligned}\tilde{g}_{\lambda,n+1}(x,y) &= e^{\lambda x}\Big(\tilde{g}_{\lambda,n}\big((2^{n+1}-1)y,y\big)\,e^{-\lambda(2^{n+1}-1)y}\\ &\quad - \frac{1}{2}\int_{(2^{n+1}-1)y}^{x}\lambda\,\tilde{g}_{\lambda,n}\big(\tfrac{t-y}{2},y\big)e^{-\lambda t}\,dt\Big),\ x\in\mathbf{R}.\end{aligned}\quad (4.4)$$

b)

$$\begin{aligned}&h_{0,p_r,\lambda}(x,y)\\ &\quad = e^{-\lambda x}\sum_{n=0}^{\infty}\tilde{h}_{0,p_r,\lambda,n}(x,y)\mathbf{1}_{[(2^n-1)y,(2^{n+1}-1)y)}(x),\end{aligned}\quad (4.5)$$

where $\tilde{h}_{0,p_r,\lambda,0}(x,y) = 0$ and

$$\begin{aligned}&\tilde{h}_{0,p_r,\lambda,n+1}(x,y)\\ &\quad = e^{\lambda x}\Big(\tilde{h}_{0,p_r,\lambda,n}\big((2^{n+1}-1)y,y\big)e^{-\lambda((2^{n+1}-1)y,y)}\\ &\qquad - \frac{1}{2}\int_{(2^{n+1}-1)y}^{x}\lambda\,\big(p_r\tilde{h}_{0,p_r,\lambda,n}(\tfrac{t-y}{2})\\ &\qquad - (1-p_r)\tilde{g}_{\lambda,n}(\tfrac{t-y}{2},y)\big)e^{-\lambda t}\,dt\Big),\ n\in\mathbf{N}.\end{aligned}$$

Proof. a) Let $\tilde{g}_{\lambda,n}(x,y) = \tilde{g}_\lambda(x,y)$ whenever $x\in[(2^n-1)y,(2^{n+1}-1)y]$, $n\in\mathbf{N}_0$. Then (2.2) may be written as a system of infinitely many inhomogenous differential equations as follows.

$$\begin{aligned}\tilde{g}_{\lambda,0}(x,y) &= e^{\lambda x}\\ \tfrac{\partial}{\partial x}\tilde{g}_{\lambda,n+1}(x,y) &= \lambda\Big(\tilde{g}_{\lambda,n+1}(x,y) - \tfrac{1}{2}\tilde{g}_{\lambda,n}\big(\tfrac{(x-y)}{2},y\big)\Big),\\ &\qquad n\in\mathbf{N},\end{aligned}\quad (4.6)$$

with successive initial conditions $\tilde{g}_{\lambda,n+1}\big((2^{n+1}-1)y,y\big) = \tilde{g}_{\lambda,n}\big((2^{n+1}-1)y,y\big)$. This type of inhomogeneous differential equation is well known to have solution (4.4) (cf. [2]). The proof of b) follows the same lines. ∎

For numerical purposes representation (4.3) allows to determine $\tilde{g}_\lambda(x,y)$ recursively in a very effective manner. This method can be easily extended to the inhomogeneous case. We offer another possibility to determine $\tilde{g}_{\lambda,n}$. Let

$$P_n(u) = \sum_{i=0}^{n}\frac{(-1)^i\,u^{2^i-1}}{2^{i(i+1)/2}m_i(1)},\quad n\in\mathbf{N}_0,\ u\in\mathbf{R}.$$

Then

$$\tilde{g}_{\lambda,n}(x,y) = \sum_{i=0}^{n} \frac{P_{n-i}\big(\exp(-\lambda y)\big)}{2^i\, m_i(1)} \exp\left(\frac{\lambda(x-(2^i-1)y)}{2^i}\right), \quad n \in \mathbf{N}.$$

We omit the complicated proof of this statement. Similarly, (4.5) admits to calculate differential equation (2.5) effectively when $y > 0$. As above, $\tilde{h}_{0,p_r,\lambda}$ may also be determined with

$$Q_{k,n,p}(u) = \sum_{i=k}^{n} \frac{(-1)^{i-k}(1-p^i)u^{(2^{i-k}-1)}}{2^{(i-k)(i-k+1)/2}\, m_{i-k}(1)}, \quad n,k \in \mathbf{N}_0,\ p \in [0,1],\ u \in \mathbf{R},$$

as

$$\tilde{h}_{0,p_r,\lambda} = -\sum_{i=1}^{n} \frac{Q_{i,n,p_r}\big(\exp(-\lambda y)\big)}{2^i\, m_i(1)} \exp\left(\frac{\lambda(x-(2^i-1)y)}{2^i}\right), \quad n \in \mathbf{N}.$$

We also skip the proof of this representation.

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