On the Optimal Base-Station Density for CDMA Cellular Networks

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***Abstract*—In this paper, the minimal base-station density for a code-division multiple-access (CDMA) cellular radio network is determined such that the outage probability does not exceed a certain threshold. Base stations are assumed to be located on a regular triangular grid of minimum distance d, while mobiles are randomly distributed according to a two-dimensional Poisson point pattern. Each mobile may be connected to, at most, one of four surrounding base stations, effectively connecting and applying power control to the one with least attenuation. Thus, we model the use of macroscopic selection diversity. We obtain a normal approximation to the total interference power at a reference base station for a correlated log-normal shadowing law. The base station distance we obtain is proportional to the inverse of the square root of the traffic intensity, and we obtain the constant of proportionality, which is itself a function of the minimum acceptable carrier-to-interference (C/I) ratio and the maximum tolerable outage probability. Our formula for this distance can be used in network planning and design.**

***Index Terms*—Base-station density, code-division multiple-access (CDMA), marked spatial Poisson point process, network planning, power control.**

I. Introduction

An IMPORTANT problem that arises in the design of a code-division multiple-access (CDMA) network is to determine how densely base transmitter stations (BTSs) need to be packed in order to carry the required traffic under certain quality constraints. Even if base stations cannot be located at the precise locations computed by our model, an answer to the above question provides valuable information about the necessary number of base station transmitters per unit area. In this paper, we provide an analytical approach to solving this problem.

As a measure of quality, we use the instantaneous outage probability at a target base station, which in turn depends on the total received interference. In a CDMA system, a mobile experiences outage when the carrier-to-interference (C/I) ratio drops below the minimum level required to meet the quality of service requirements of the mobile. In this paper, we assume that all the users require the same quality of service and hence are received at the same power level.

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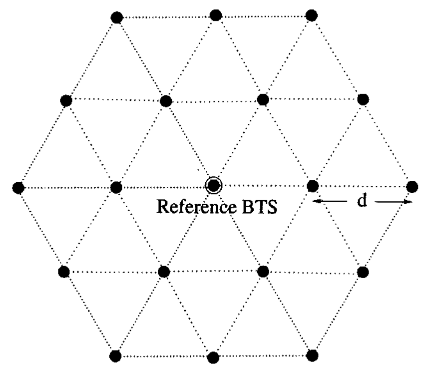

Fig. 1. Grid of base station locations.

In Section II, we take a regular, infinite triangular grid of base stations in the plane, with grid constant d being the distance between any two base stations, as depicted in Fig. 1. Mobiles are assumed to be scattered randomly in the plane according to a homogeneous two-dimensional Poisson point pattern (process) with constant intensity λ [8].

Our interest is in the probability of outage at a reference base station. We assume that a user is received with unit power at the base station it actually connects to, and that outage occurs if the C/I ratio at the base station drops below a threshold α. It is argued in [13] that "instantaneous outage probability" is an adequate performance measure for network dimensioning. In this paper, we have built this assumption into the model, since no time dynamics are considered.

We assume the traffic intensity λ, the C/I ratio threshold α, and the target outage probability μ, are available as design parameters, and we obtain (20) for the maximum grid distance d, and hence, the minimal base-station density, that can achieve these parameters. The focus in this paper is on solving for the minimum base-station density. However, if instead the grid distance d is fixed, (20) can be solved to give the maximum traffic intensity λ.

Our model for how mobiles create interference at the reference base station is based on the power control and macroscopic selection diversity used in IS-95 CDMA (see [15]). Specifically, we assume that each user can potentially be connected to any of the three closest base stations, or to the reference base station, as depicted in Fig. 2. The mobile actually connects to the base station for which the attenuation is least. The mobile employs power control so that the received power at the connected station is unity, and it follows that the interference it creates at the reference base station is upper bounded by unity, with equality if and only if it connects to the reference base station.

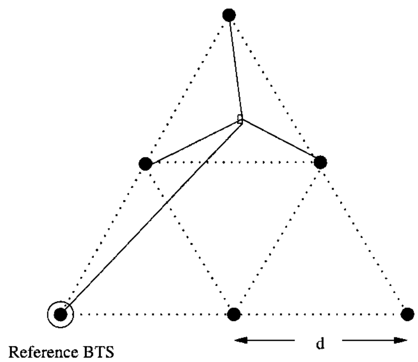

Fig. 2. One large triangle: One-sixth of the hexagon.

The model we use for power control is slightly different from that implemented in current CDMA systems. In IS-95, target received powers at the base stations vary according to multipath effects, which are measured via the frame error rate at the base station. The dynamics are quite complex, in that frame error rates also depend on the received interference, which actually couples the power control between users [6]. The IS-95 algorithm was designed for voice systems, where the mobiles can be moving at high speed and the algorithm adjusts powers rapidly in response to both fading and interference [14]. The objective is to try to avoid outage, if at all possible.

In this paper, we take the simpler model of fixed unit target received power at the connected base station, rather than coupling the power control between users. This assumption has been made in many other works [3], [5], [15], [17] (among others), and we are motivated by a number of considerations here. First, as with these other papers, we want a tractable model, and particularly one that does not allow interactions between users, since the theory we are using cannot account for that. Second, we want our method to be applicable to future bursty data systems, and in these systems, it may not be appropriate to try and avoid outage through rapid power control. Instead, outage (loss) can be tolerated since packets can be retransmitted, and the objective then is to keep the outage probability within a bound, which is chosen to avoid excessive delay in the retransmissions of lost packets.

In this paper, we do not consider multipath effects, but we remark that the marginal distribution of multipath fading can be introduced in exactly the same way that we model log-normal shadowing, if desired. The distribution of total interference power at a reference station, taking account of Ricean and Rayleigh fading as well as log-normal shadowing, has been determined in [7] and [9]. It is certainly of interest to extend our results to handle random target received power requirements, as in [13], [14], and [16], as these may vary with the channel multipath conditions. It may also be of interest to consider imperfect power control [13], [14]. Such extensions are straightforward, but we do not consider them here.

Part of the paper is concerned with the task of taking into account the effect of the cumulative interference from a large number of mobiles, and this has been the topic of many other research works. Our particular concern has been to obtain a tractable *analytical* model that can then be used in the *optimization* of base-station density. We also believe it may prove useful in other optimization problems, such as occur in the pricing of spectrum. One way to meet increasing capacity demand is to obtain more spectrum, the other is to place more base stations, and a study of the tradeoff between the two is of interest.

In [15], a *numerical* method to characterize the total interference was provided, under similar assumptions of uniform traffic pattern, regular triangular grid of base stations, log-normal shadowing, macroscopic selection diversity, and power control. However, in [15], mobiles are not taken to be discrete entities, but are uniformly smeared over the service area, and only mean interference levels are calculated. In this paper, we assume instead a uniform spatial Poisson point pattern to describe the mobile locations.

Earlier work on interference characterization arising from mobiles distributed spatially as a Poisson point pattern appeared in [10]–[12]. The results there have a different character from our results. In these works, there is no cellular structure and hence, no macroscopic selection diversity or power control. On the other hand, the model of the physical layer is more detailed, including fading, spreading, and modulation.

Reference [10] is particularly related, since it also attempts to find an optimal transmission distance. In [10], an ad-hoc network scenario is considered, where data packets are transmitted over multiple hops. The aim is to obtain the optimal transmission range from one node to the next. The formula obtained therein for the optimal transmission range bears some similarity to our expression for the minimum distance between base stations (20). Not surprisingly, both scale inversely with the square root of the traffic intensity. However, in our case, there is a term in the constant that scales with the square root of the processing gain, whereas in [10], the constant scales with the 0.25 power of the processing gain. This difference is likely to be due to the multiple transmissions required in the multihop network case.

A complete characterization of the distribution function from a user to a reference base station is obtained in [17], taking into account distance, shadow attenuation, and power control. The traffic model presupposes that users in each cell are uniformly distributed in a circle around the respective base station. The distribution of interference power is expressed in terms of a multiple integral which can be evaluated numerically. From this, numerical values of the outage probability in time-division multiple-access (TDMA)- and CDMA-like cellular systems are calculated.

Another related paper is [13], which considers the coverage-capacity tradeoff, but the focus is on a single-cell scenario. This paper provides an extension to the multicell scenario.

II. Interference Modeling

As discussed in the introduction, we assume that mobiles are located in the plane according to a homogeneous, spatial Poisson point pattern of intensity λ ([8]), and base stations are located in an infinite triangular grid, with grid distance d, as depicted in Fig. 1. Let Π denote the set of random mobile locations in the plane, and if a mobile happens to be located at position $\boldsymbol{x}$ in the plane, let $I(\boldsymbol{x})$ denote the random interference created at the reference base station, whether or not the mobile actually connects to this base station. Then the total interference is given by $\sum_{\boldsymbol{X}\in\Pi} I(\boldsymbol{X})$.

To completely specify the total interference statistics, we need to specify the statistics of $I(\boldsymbol{x})$ at any point $\boldsymbol{x}$ in the plane. Note that by assumption, the interference levels at the reference base station from the mobiles are independent of each other, so the marginal statistics of each mobile are sufficient to characterize the marginal statistics of the total interference.

To cut down on the numerics, we will neglect interference from mobiles sufficiently far from the reference base station. Adjacent to the reference base station are six neighboring base stations, and beyond that, twelve base stations that are "one hop" away (Fig. 1). The total region contained in the convex hull of these twelve second-tier base stations forms a hexagon, and we assume that mobiles outside of this hexagon contribute negligible interference at the reference base station. Thus, we effectively restrict the spatial point pattern of mobiles to this hexagon, as far as our analysis is concerned, although in principle, our techniques can be applied to a region of arbitrary size and shape.

By rotational symmetry, we can restrict attention further to any one of six triangles that make up this hexagon, as depicted in Fig. 2, where the triangle is bounded by six base stations as depicted there, with the reference base station at a vertex. This triangle is further subdivided into four smaller triangles, each defined by three adjacent base stations that are the vertices of the smaller triangles. A mobile located in one of these smaller triangles can potentially connect to any of the three base stations at the vertices of the mobile's triangle, plus the reference base station. If the mobile falls in the lower left triangle in the figure, a "first-tier" triangle, then the macroscopic selection diversity is limited to three base stations, since the reference base station is one of the three adjacent base stations. However, if it lies in one of the three "second-tier" triangles, the macroscopic selection diversity includes the reference base station.

To determine the interference from a location in the first-tier (lower left) triangle, let $\boldsymbol{x}$ denote the position of a mobile relative to a coordinate system with origin at the reference base station, and orientation as depicted in Fig. 3. Let $d_0(\boldsymbol{x}), d_1(\boldsymbol{x}), d_2(\boldsymbol{x})$ be the distances to the adjacent base stations, with 0 denoting the reference base station. Then

$$\begin{aligned} d_0(\boldsymbol{x}) =& x_1^2 + x_2^2 \\ d_1^2(\boldsymbol{x}) =& \left(x_1 - \frac{d}{2}\right)^2 + \left(x_2 - \frac{\sqrt{3}d}{2}\right)^2 \\ d_1^2(\boldsymbol{x}) =& \left(x_1 + \frac{d}{2}\right)^2 + \left(x_2 - \frac{\sqrt{3}d}{2}\right)^2. \end{aligned} \quad (1)$$

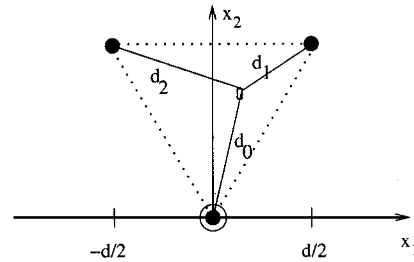

Fig. 3. Orientation of first-tier coordinate system.

We assume the typical path-loss model, in which the signal strength degrades as an inverse power law, with exponent γ, such that the attenuation at a distance d is given by $d^{-\gamma}$. For mobile communication environments γ is typically in the order of 3 to 5. Furthermore, the signal is also multiplied by a log-normal random variable, representing shadow fading. The way we model this is to let L_0, L_1, L_2 denote independent random variables, with

$$L_i = 10^{G_i/10},\ G_i \sim N\left(0, \sigma^2\right),\ i = 0, 1, 2.$$

Experiments suggest the choice of $\sigma = 8$ for standard deviation of G_i (cf. [15]). As in [15], we assume that G_0, G_1, G_2 are jointly Gaussian. Then we assume that the mobile located at position x in the figure is received at the reference base station with power $I(\boldsymbol{x})$, as shown in the equation at the bottom of the page. This definition encompasses the effect of path loss, shadowing, and also that of macroscopic selection diversity and power control.

Let $H_i = G_0 - G_i$, $i = 1, 2$. It holds that
$(H_1, H_2)' = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} (G_0, G_1, G_2)'$
such that $(H_1, H_2)'$ is jointly Gaussian, with mean $(0, 0)'$ and covariance matrix
$\Sigma = \sigma^2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.
The joint density of $(H_1, H_2)'$ is given by

$$\begin{aligned} f(h_1, h_2) =& \frac{1}{2\pi} |\Sigma|^{-1/2} \exp\left(-\frac{1}{2}\boldsymbol{x}'\Sigma^{-1}\boldsymbol{x}\right) \\ =& \frac{1}{2\pi\sqrt{3}\sigma^2} \exp\left(\frac{-\left(h_1^2 - h_1 h_2 + h_2^2\right)}{(3\sigma^2)}\right). \end{aligned}$$

With a view toward extending to second-tier triangles, where we will need to include more shadowing random variables and alternative coordinate systems, let us now include in the notation the random shadowing effects as parameters, together with the

$$I(\boldsymbol{x}) = \begin{cases} 1, & \text{if } d_0^{-\gamma}(\boldsymbol{x})L_0 > d_1^{-\gamma}(\boldsymbol{x})L_1 \text{ and } d_0^{-\gamma}(\boldsymbol{x})L_0 > d_2^{-\gamma}(\boldsymbol{x})L_2 \\ \dfrac{d_0^{-\gamma}(\boldsymbol{x})L_0}{d_1^{-\gamma}(\boldsymbol{x})L_1}, & \text{if } d_1^{-\gamma}(\boldsymbol{x})L_1 > d_0^{-\gamma}(\boldsymbol{x})L_0 \text{ and } d_1^{-\gamma}(\boldsymbol{x})L_1 > d_2^{-\gamma}(\boldsymbol{x})L_2 \\ \dfrac{d_0^{-\gamma}(\boldsymbol{x})L_0}{d_2^{-\gamma}(\boldsymbol{x})L_2}, & \text{if } d_2^{-\gamma}(\boldsymbol{x})L_2 > d_0^{-\gamma}(\boldsymbol{x})L_0 \text{ and } d_2^{-\gamma}(\boldsymbol{x})L_2 > d_1^{-\gamma}(\boldsymbol{x})L_1 \end{cases}$$

coordinates of the reference base station, which in this case sits at the origin. Then $I(\boldsymbol{H},\boldsymbol{x},\mathbf{0})$ may be written as

$$\begin{aligned} I(\boldsymbol{H},\boldsymbol{x},\mathbf{0}) &= \min\left\{1, \frac{d_0^{-\gamma}(\boldsymbol{x})}{d_1^{-\gamma}(\boldsymbol{x})}10^{H_1/10}, \frac{d_0^{-\gamma}(\boldsymbol{x})}{d_2^{-\gamma}(\boldsymbol{x})}10^{H_2/10}\right\} \\ &= \min\left\{1, \left(\frac{d_0^2(\boldsymbol{x})}{d_1^2(\boldsymbol{x})}\right)^{-\gamma/2} 10^{H_1/10}, \right. \\ &\qquad \left. \left(\frac{d_0^2(\boldsymbol{x})}{d_2^2(\boldsymbol{x})}\right)^{-\gamma/2} 10^{H_2/10}\right\} \end{aligned} \tag{2}$$

with the above joint density of H_1 and H_2.

Denote the first-tier triangle by $\triangle_f$ and let the Poisson point pattern of mobiles that are located in this triangle be denoted by $\Pi_{\triangle_f}$. We can then write

$$I_{(\mathbf{0})} = \sum_{\boldsymbol{X}\in\Pi_{\triangle_f}} I(\boldsymbol{H},\boldsymbol{X},\mathbf{0}) \tag{3}$$

for the total interference at the reference base station due to the first-tier triangle.

Theorem 1: Under the above assumptions, the mean and variance of the total interference from each of the triangles in the first tier is given by

$$E\left(I_{(\mathbf{0})}\right) = \lambda d^2 c_1(\mathbf{0},\sigma) \tag{4}$$

$$V\left(I_{(\mathbf{0})}\right) = \lambda d^2 c_2(\mathbf{0},\sigma) \tag{5}$$

where $c_k(\mathbf{0},\sigma)$ for $k = 1,2$ is given by (6) at the bottom of the page, and ${d'_0}^2(\boldsymbol{x}) = x_1^2 + x_2^2$, ${d'_1}^2(\boldsymbol{x}) = (x_1 - 1/2)^2 + (x_2 - \sqrt{3}/2)^2$, and ${d'_2}^2(\boldsymbol{x}) = (x_1 + 1/2)^2 + (x_2 - \sqrt{3}/2)^2$ are obtained from (1) by setting $d = 1$.

Proof: We consider the expression for $I_{(\mathbf{0})}$ in (3), which we note is a functional of a marked Poisson point pattern ([8]). Thus, by Campbell's theorem [8],

$$E\left[I_{(\mathbf{0})}\right] = \int\int_{\triangle_f} E_{\boldsymbol{H}}\left[I(\boldsymbol{H},\boldsymbol{x},\mathbf{0})\right]\lambda d\boldsymbol{x}, \tag{7}$$

$$V\left[I_{(\mathbf{0})}\right] = \int\int_{\triangle_f} E_{\boldsymbol{H}}\left[I^2(\boldsymbol{H},\boldsymbol{x},\mathbf{0})\right]\lambda d\boldsymbol{x} \tag{8}$$

where $E_{\boldsymbol{H}}$ denotes the expectation with respect to the random vector $\boldsymbol{H}$. By (7)

$$E\left[I_{(\mathbf{0})}\right] = \lambda\int_0^{\sqrt{3}/2d}\int_{-(\sqrt{3}/3)x_2}^{(\sqrt{3}/3)x_2} E_{\boldsymbol{H}}\left(I(\boldsymbol{H},\boldsymbol{x},\mathbf{0})\right)dx_1dx_2.$$

Substituting $I(\boldsymbol{H},\boldsymbol{x},\mathbf{0})$ from (2) and changing variables $d\cdot x_i \to x_i$, we obtain (4). Equation (5) is derived from (8) along the same lines. □

We now consider the interference created from the three triangles in the second tier. There are really two distinct triangles to be considered here, which we refer to as type 1 and type 2 triangles, as depicted in Fig. 4. Both triangles of type 1 are identical as far as the statistics of interference at the reference base station are concerned. For each type of triangle, we assign an appropriate coordinate system, as depicted in Fig. 4, and then the only difference between the triangles is the location of the reference base station in the respective coordinate systems. Thus, we can reduce the problem to that of specifying the interference at an arbitrary base station location $\boldsymbol{b} = (b_1,b_2)'$, from the canonical triangle and coordinate system depicted in Fig. 5.

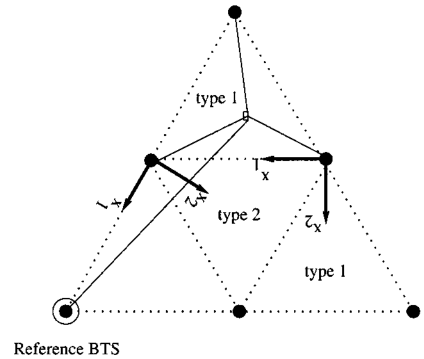

Fig. 4. Second-tier triangles with corresponding coordinate systems.

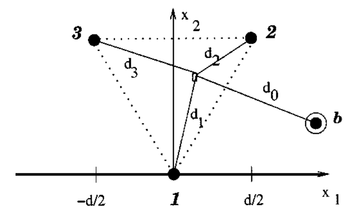

Fig. 5. Orientation of the second-tier coordinate system.

In Fig. 5, the reference base station is located at the point $\boldsymbol{b}$ and we assume that $\boldsymbol{b} \neq \mathbf{0}$, since the origin is now the location of the base station denoted by 1 in the figure. To determine the interference received at $\boldsymbol{b}$ from position $\boldsymbol{x}$ in the triangle, we now need four independent log-normal random variables $L_i, i = 0,1,2,3$. We assume that the four depicted base stations, including the reference base station located at point $\boldsymbol{b}$, employ macroscopic selection diversity. The variable L_0 denotes the shadowing to the reference base station. Further, let $\boldsymbol{H} = (H_1,H_2,H_3)$ with $H_i = L_0 - L_i$, $i = 1,2,3$. Then the received power at $\boldsymbol{b}$, conditional on $\boldsymbol{H}$ and the mobile being at $\boldsymbol{x}$, is given by

$$\begin{aligned} I(\boldsymbol{H},\boldsymbol{x},\boldsymbol{b}) = \min\Bigg\{1, &\left(\frac{d_0^2(\boldsymbol{x})}{d_1^2(\boldsymbol{x})}\right)^{-\gamma/2} 10^{H_1/10}, \\ &\left(\frac{d_0^2(\boldsymbol{x})}{d_2^2(\boldsymbol{x})}\right)^{-\gamma/2} 10^{H_2/10}, \\ &\left(\frac{d_0^2(\boldsymbol{x})}{d_3^2(\boldsymbol{x})}\right)^{-\gamma/2} 10^{H_3/10}\Bigg\} \end{aligned} \tag{9}$$

$$c_k(\mathbf{0},\sigma) = \int_0^{\sqrt{3}/2}\int_{-(\sqrt{3}/3)x_2}^{(\sqrt{3}/3)x_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(h_1,h_2)\min\left\{1, \left(\frac{{d'_0}^2(\boldsymbol{x})}{{d'_1}^2(\boldsymbol{x})}\right)^{-\gamma/2} 10^{h_1/10}, \left(\frac{{d'_0}^2(\boldsymbol{x})}{{d'_2}^2(\boldsymbol{x})}\right)^{-\gamma/2} 10^{h_2/10}\right\}^k dh_1dh_2dx_1dx_2 \tag{6}$$

where the distances d_0, d_1, d_2, d_3 are given by

$$\begin{aligned} d_0^2(\boldsymbol{x}) &= (x_1 - b_1)^2 + (x_2 - b_2)^2 \\ d_1^2(\boldsymbol{x}) &= x_1^2 + x_2^2 \\ d_2^2(\boldsymbol{x}) &= \left(x_1 - \frac{d}{2}\right)^2 + \left(x_2 - \frac{\sqrt{3}d}{2}\right)^2 \\ d_3^2(\boldsymbol{x}) &= \left(x_1 + \frac{d}{2}\right)^2 + \left(x_2 - \frac{\sqrt{3}d}{2}\right)^2 \end{aligned} \tag{10}$$

in terms of the coordinate system of Fig. 5.

The vector $\boldsymbol{H}$ is jointly Gaussian, with expectation zero and covariance matrix

$$\boldsymbol{\Sigma} = \sigma^2 \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The joint density is

$$f(h_1, h_2, h_3) = \frac{1}{(2\pi)^{3/2} 2\sigma^3} \exp\left(-\frac{1}{8\sigma^2} (h_1, h_2, h_3) \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \right).$$

Let $\triangle$ denote an arbitrary triangle in the second tier of triangles, and let $\Pi_\triangle$ denote the Poisson point pattern of mobiles that are located in this triangle. Then

$$I_{(\boldsymbol{b})} = \sum_{\boldsymbol{X} \in \Pi_\triangle} I(\boldsymbol{H}, \boldsymbol{X}, \boldsymbol{b}) \tag{11}$$

where $I_{(\boldsymbol{b})}$ denotes the interference at the reference base station located at $\boldsymbol{b}$ from the mobiles in the triangle $\triangle$. The mean and variance of $I_{(\boldsymbol{b})}$ are given in the following theorem, which we state without proof, since the proof follows almost identically to that of *Theorem 1*.

Theorem 2: Under the above assumptions, with $\boldsymbol{b} \neq \boldsymbol{0}$, the mean and variance of the total interference at the reference base station, located at point $\boldsymbol{b}$, from the mobiles in the triangle depicted in Fig. 5, are given by

$$E\left(I_{(\boldsymbol{b})}\right) = \lambda d^2 c_1(\boldsymbol{\beta}, \sigma) \tag{12}$$

$$V\left(I_{(\boldsymbol{b})}\right) = \lambda d^2 c_2(\boldsymbol{\beta}, \sigma) \tag{13}$$

where $c_k(\beta, \sigma)$ for $k = 1, 2$, see (14) at the bottom of the page. $\boldsymbol{b} = d \cdot \boldsymbol{\beta}$ and ${d'_0}^2(\boldsymbol{x}) = (x_1 - \beta_1)^2 + (x_2 - \beta_2)^2$, ${d'_1}^2(\boldsymbol{x}) = x_1^2 + x_2^2$, ${d'_2}^2(\boldsymbol{x}) = (x_1 - 1/2)^2 + (x_2 - \sqrt{3}/2)^2$, and ${d'_3}^2(\boldsymbol{x}) = (x_1 + 1/2)^2 + (x_2 - \sqrt{3}/2)^2$ are obtained from (10) by setting $d = 1$.

TABLE I
CONSTANTS FOR FIRST- AND SECOND-TIER TRIANGLES

σ	0	2	4	6	8	10
$c_1(\boldsymbol{\beta}_1, \sigma)$	0.00220	0.00250	0.00361	0.00633	0.0111	0.0173
$c_1(\boldsymbol{\beta}_2, \sigma)$	0.00716	0.00813	0.0116	0.0182	0.0263	0.0345
$c_1(\boldsymbol{0}, \sigma)$	0.2817	0.2391	0.2165	0.2034	0.1951	0.1895
$c_2(\boldsymbol{\beta}_1, \sigma)$	$2.172 \cdot 10^{-5}$	$3.810 \cdot 10^{-5}$	0.000197	0.00136	0.00468	0.00986
$c_2(\boldsymbol{\beta}_2, \sigma)$	0.000260	0.000459	0.00201	0.00690	0.0144	0.0226
$c_2(\boldsymbol{0}, \sigma)$	0.2560	0.1950	0.1775	0.1709	0.1679	0.1663

We can now apply *Theorem 2* to obtain the mean and variance of interference from the two different types of second-tier triangles. For type 1 triangles, we set $\boldsymbol{b}_1 = \left(3d/2, \sqrt{3}d/2\right)$ and hence, $\boldsymbol{\beta}_1 = \left(3/2, \sqrt{3}/2\right)$. For type 2 triangles, we set $\boldsymbol{b}_2 = (d, 0)$ and hence, $\boldsymbol{\beta}_2 = (1, 0)$. The corresponding coordinate systems are indicated by boldface arrows in Fig. 4. Obviously, there are other equivalent choices of coordinate transforms leading to the same results.

III. APPROXIMATION FOR OUTAGE PROBABILITY

Applying *Theorems 1* and *2* to the first- and second-tier triangles, we obtain the mean and variance of the total received power I at the reference base station from all mobiles in the first and second tier

$$E(I) = \lambda d^2 c_E \tag{15}$$

$$V(I) = \lambda d^2 c_V \tag{16}$$

where the constants c_E and c_V are given by

$$c_E = 12 c_1(\boldsymbol{\beta}_1, \sigma) + 6 c_1(\boldsymbol{\beta}_2, \sigma) + 6 c_1(\boldsymbol{0}, \sigma) \tag{17}$$

$$c_V = 12 c_2(\boldsymbol{\beta}_1, \sigma) + 6 c_2(\boldsymbol{\beta}_2, \sigma) + 6 c_2(\boldsymbol{0}, \sigma). \tag{18}$$

To obtain particular values, we use Table I, which has been computed from (6) and (14) via Monte Carlo integration.

Outage occurs whenever the C/I ratio of a mobile station falls short of a minimum threshold α. Hence, the instantaneous outage probability is given by $P(I > 1/\alpha)$. Limiting this probability by a (typically small) value μ leads to

$$P\left(I > \frac{1}{\alpha}\right) \leq \mu. \tag{19}$$

Since we have expressions for the mean and variance of I, it is very natural to approximate I with a Normal random variable. Let $u_{1-\mu}$ denote the $1-\mu$-quantile of the standard normal distribution, i.e., the unique solution of the equation $\Phi(u_{1-\mu}) = 1 - \mu$, where $\Phi(x) = \left(1/\sqrt{2\pi}\right) \int_{-\infty}^{x} \exp\left(-t^2/2\right) dt$ denotes the standard normal distribution function. Applying the normal approximation, (19) is transformed to

$$d^2 \lambda c_E + d u_{1-\mu} \sqrt{c_V \lambda} \leq \frac{1}{\alpha}.$$

$$\begin{aligned} c_k(\boldsymbol{\beta}, \sigma) = \int_0^{\sqrt{3}/2} \int_{-(\sqrt{3}/3)x_2}^{(\sqrt{3}/3)x_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(h_1, h_2, h_3) \\ \times \min\left\{ 1, \left(\frac{{d'_0}^2(\boldsymbol{x})}{{d'_1}^2(\boldsymbol{x})}\right)^{-\gamma/2} 10^{h_1/10}, \left(\frac{{d'_0}^2(\boldsymbol{x})}{{d'_2}^2(\boldsymbol{x})}\right)^{-\gamma/2} 10^{h_2/10}, \left(\frac{{d'_0}^2(\boldsymbol{x})}{{d'_3}^2(\boldsymbol{x})}\right)^{-\gamma/2} 10^{h_3/10} \right\}^k dh_1 dh_2 dh_3 dx_1 dx_2 \end{aligned} \tag{14}$$

The maximum d satisfying this inequality is given by

$$d^* = \frac{1}{\sqrt{\lambda}} \frac{1}{2c_E} \cdot \left(\sqrt{u_{1-\mu}^2 c_V + \frac{4c_E}{\alpha}} - u_{1-\mu}\sqrt{c_V} \right). \quad (20)$$

Equation (20) gives a simple rule for how the base-station density in a regular hexagonal pattern should be chosen in such a way as to ensure that the maximal outage probability μ is not exceeded. Note that the base-station distance d^* decreases as $\lambda^{-1/2}$, up to a constant dependent on the minimal C/I ratio α and the outage probability μ.

We remark that the Normal approximation we have used here is standard in the literature [4], [5]. A formal justification based on the central limit theorem is provided in Section 6.1 of [4], and a very similar argument applies here. We remark that refinements to the central limit theorem have been considered in [1] and [2] for a similar spatial Poisson traffic model of interferers. The results in Section IV certainly support the approximation for the problem under investigation in this paper.

IV. Validation of the Approximation

We can consider (20) as an approximation to find the true optimal value of d. Figs. 6–8 contain plots for $\alpha = 0.01$, $\alpha = 0.05$, and $\alpha = 0.1$, respectively. In all three cases, the intensity λ is equal to unity, $\sigma = 8$, and $\gamma = 4$. The outage probability μ is a free parameter, depicted on the x axis of the graphs. The logarithm used for the x axis is $\log_{10}$. The y axis depicts the grid distance d as predicted by the model.

To check the accuracy of our approximation, we also plot the results of Monte Carlo simulations, in which many trials are performed and, in each trial, mobile locations are generated randomly from a Poisson point pattern of unit intensity over a large region. The region we consider is a circle of radius $5d$, which is more than large enough to capture all interference effects at the reference base station. We measure the empirical outage probability for different values of d, and conduct enough trials in each case to ensure that the estimates of the standard deviation are appropriately small. We require that the outage probability estimate is accurate to the number of significant figures in the estimate, which is two, except in the case of outage probabilities less than 0.01, where we only obtain one significant figure.

We observe from the graphs that the approximation is accurate over a wide range of α and over a wide range of outage probabilities. An interesting feature is that the optimal d^* is not very sensitive to the outage probability, as the latter quantity ranges from 0.001 to 0.5. The sensitivity to μ decreases as α gets smaller. This suggests that the choice of μ is not that critical in obtaining a rough estimate of d^*. Conversely, the outage probability is very sensitive to d, implying that a conservative choice for d is required, rather than one accurately computed from a particular model.

Note that a "simulation" curve taken from any one of the figures is actually a kind of "universal curve", in that results for any λ can be obtained by scaling d appropriately. This follows since traffic density can always be normalized to 1, by measuring distance on an appropriate scale. This fact is also reflected in the approximation (20), in that d^* is inversely proportional to $\sqrt{\lambda}$.

Why not just use such a universal curve in the design of a cellular network? One answer is that a lot of simulation time goes into constructing such a curve, and it is not then robust to changes in the model. This is an important consideration, since in practice there are many different fading models, appropriate to different scenarios, and many scenarios may be encountered in a single network design. We have considered only a highly idealized model of path loss in the present paper. Indeed, the appropriate fading model, and the parameters of that model, may depend on how close together base stations are placed, and it may be necessary to undertake an iterative approach to design, in which a computed d determines a model and vice versa, allowing successive refinements of the model. Note that in our

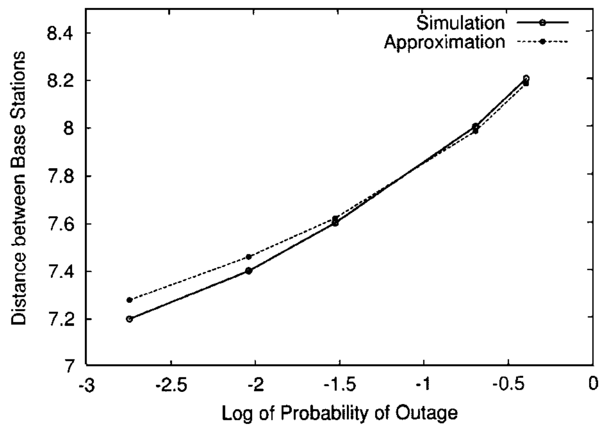

Fig. 6. $\alpha = 0.01, \lambda = 1, \sigma = 8, \gamma = 4$.

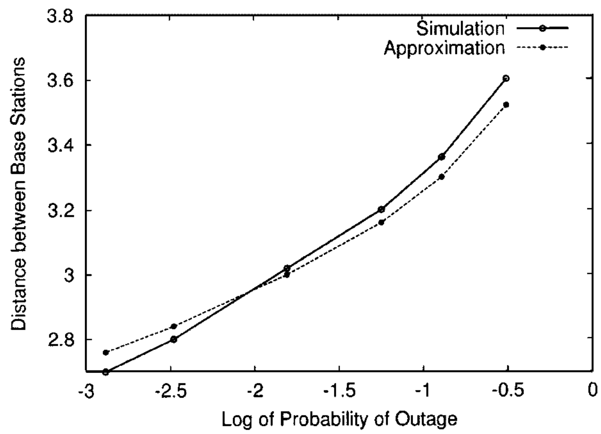

Fig. 7. $\alpha = 0.05, \lambda = 1, \sigma = 8, \gamma = 4$.

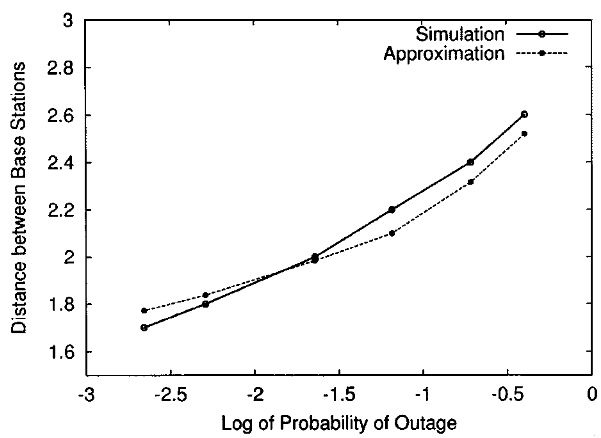

Fig. 8. $\alpha = 0.1, \lambda = 1, \sigma = 8, \gamma = 4$.

approximation, changing the model only requires recomputing the constants c_E and c_V. We remark that the final design may require a more detailed simulation-based approach. Another very important advantage of our method is that (20) is an analytical formula, once the constants c_E and c_V have been evaluated. As such, it can be used as the basis of further analytical work, such as in the pricing of spectrum.

V. Conclusion

The main intention of this paper is to provide a method to obtain a simple rule for determining the minimal base-station density on a regular triangular grid. The requirement is to maintain a given maximum outage probability μ for spatially uniform Poisson traffic of intensity λ. It turns out that the optimal grid constant d is proportional to $1/\sqrt{\lambda}$, with a constant depending on the outage probability μ and the amount of interference at which outage occurs $1/\alpha$. Our results are based on a geometrical analysis of the network with mobiles distributed according to a two-dimensional Poisson point pattern and a normal approximation to the complicated distribution of total interference power at a reference base station. Power control, log-normal shadowing, and macroscopic selection diversity effects are all taken into account. The relationship between d, λ, μ, and α is given in (20).

Figs. 6–8 suggest that (20) provides a very good first-order approximation for obtaining d. We observe that it works well over a wide range of outage probabilities and of C/I ratio targets. One extra insight we obtain from the results is that the outage probability is very sensitive to the distance between base stations, which suggests a conservative approach to placing base stations is required.

It is of interest in future work to see how robust the method is to variation of the parameters of the propagation model and to extend the model itself by small-scale fading. Other works have considered cumulated interference power under Rayleigh fading, for example, see [7]. The level of accuracy we have obtained in the present paper is enough for a design procedure in which the first goal is to obtain orders of magnitude, after which much more precise propagation modeling is used to find the final locations of base stations, taking into account the locations of physical features and the sites which might provide base station locations.

Our model is for a spatially homogenous Poisson point pattern of mobiles in the plane. It is straightforward to obtain analogous results for a uniform Poisson point pattern on the line, which may be useful for modeling mobiles along roads or highways. A more challenging direction is to try and relax the assumption of spatial homogeneity, to allow for a higher density of base stations in areas of traffic hotspots. This is a topic for future work.

Appendix

One assumption that may need to be reconsidered in future work is our assumption that the macroscopic selection diversity always includes the reference base station. This assumption implies that as soon as a mobile creates a large amount of interference at a base station, it instantaneously switches this base station into its macroscopic selection diversity active set, thereby reducing the interference it creates to no more than unity. This assumption is sometimes made (e.g., [17]) but may not be realistic, depending on how long it takes to acquire a new base station signal and measure its power on the pilot channel. Its validity also depends on whether or not there is a limit on how many base stations a mobile can be connected to simultaneously. It is of interest to examine the effect of relaxing this assumption. To get some idea of the effect, albeit with a fairly crude model, we include here results for the scenario in which mobiles in type 1 triangles never power control to the reference base station. In this way, we observe the effect of the heavy tail of log-normal shadowing, which does not have any impact under the assumptions made in the main body of this paper, since power control is then assumed to eliminate the effect.

TABLE II
CONSTANTS FOR FIRST AND SECOND TIER TRIANGLES, WITH REFERENCE BASE EXCLUDED FOR TYPE 1 TRIANGLES

σ	0	2	4	6	8	10
$c_1(\boldsymbol{\beta}_1,\sigma)$	0.00220	0.00250	0.00361	0.00633	0.0156	0.0482
$c_1(\boldsymbol{\beta}_2,\sigma)$	0.00716	0.00813	0.0116	0.0182	0.0263	0.0345
$c_1(\mathbf{0},\sigma)$	0.2817	0.2391	0.2165	0.2034	0.1951	0.1895
$c_2(\boldsymbol{\beta}_1,\sigma)$	$2.174\cdot10^{-5}$	$3.812\cdot10^{-5}$	0.000199	0.00298	0.115	11.4
$c_2(\boldsymbol{\beta}_2,\sigma)$	0.000260	0.000459	0.00201	0.00690	0.0144	0.0226
$c_2(\mathbf{0},\sigma)$	0.2560	0.1950	0.1775	0.1709	0.1679	0.1663

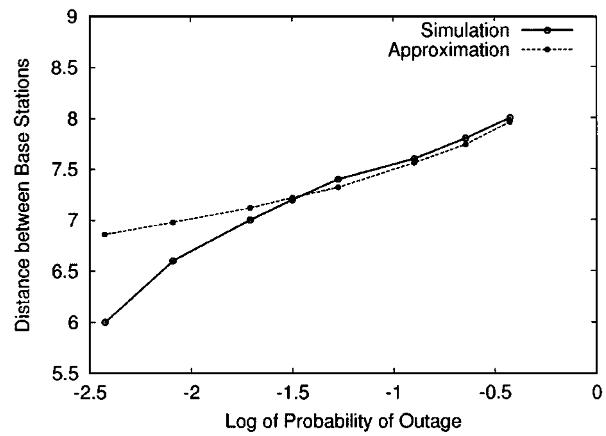

Fig. 9. $\alpha = 0.01, \lambda = 1, \sigma = 8, \gamma = 4$.

The revised table of constants under the new assumption is given in Table II. There is very little difference between this table and Table I, apart from the $c_1(\boldsymbol{\beta}_1,\sigma)$ term at $\sigma = 10$ and the $c_2(\boldsymbol{\beta}_1,\sigma)$ terms at $\sigma \geq 6$. Thus, for the usual case of $\sigma = 8$, it is the constants associated with the variance of the interference that are significantly different under the new assumption.

We have repeated the calculations of d under the same scenarios as examined in Section IV. Thus, Figs. 9–11 contain plots for $\alpha = 0.01$, $\alpha = 0.05$, and $\alpha = 0.1$, respectively. In all three cases, the intensity λ is equal to unity, $\sigma = 8$, and $\gamma = 4$. Again, the outage probability μ is a free parameter, depicted on the x axis of the graphs. The y axis depicts the grid distance d as predicted by the model.

The corresponding Monte Carlo simulation results depicted in the graphs now assume that only users in the first-tier triangles or the second tier, type 2 triangles, include the reference base station in the macroscopic selection diversity. All other users ranging out to a distance $5d$ from the reference base station only

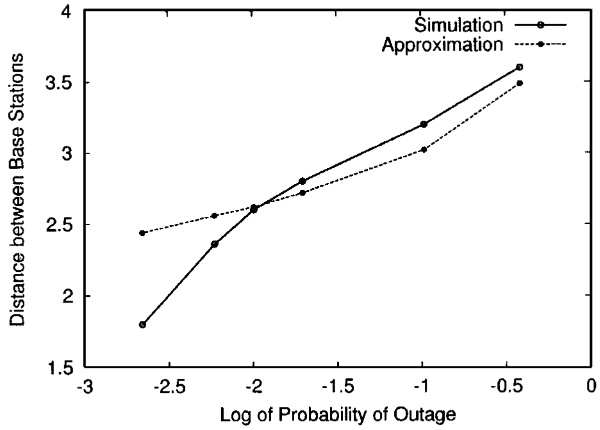

Fig. 10. $\alpha = 0.05, \lambda = 1, \sigma = 8, \gamma = 4$.

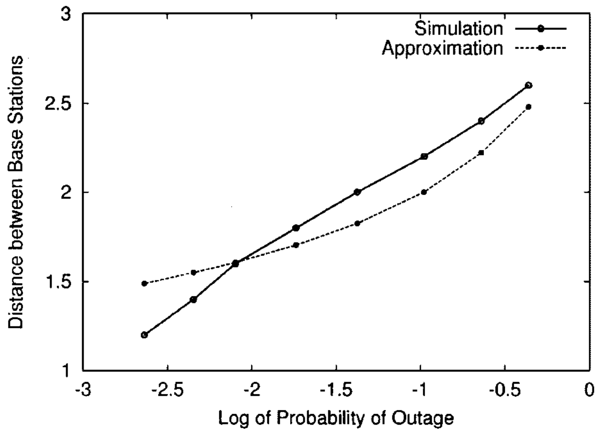

Fig. 11. $\alpha = 0.1, \lambda = 1, \sigma = 8\ \gamma = 4$.

power control to the three adjacent base stations and can create arbitrarily high interference at the reference base station.

We observe that the approximation (20) is still reasonably accurate for outage probabilities above 0.01, and at larger outage probabilities gives a conservative choice for d^*. However, the model clearly breaks down for very small outage probabilities and gives distances that are too large to support the traffic at those low levels of outage probability. A truncation technique, used in the appendix of [16], may prove useful in these scenarios, but this is a topic for future research.

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