Iterated Water-Filling for OFDMA Rate and Power Allocation with Proportionality Constraints

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***Abstract*—The present correspondence deals with the rate and power allocation problem for multi-user orthogonal frequency division multiple access (OFDMA). Using directional derivatives we first derive an explicit solution of the single-user OFDM power allocation problem for a general class of rate-power functions. In a nested algorithm, this solution is used to determine the solution of the closely related rate allocation problem for both the single-user and multi-user case with proportionality constraints. The results are applied to a widely used class of rate-power function.**

***Index Terms*—directional derivatives, generalized water-filling, multi-user channels, OFDM**

I. Introduction

Orthogonal frequency division multiplexing (OFDM) is a modulation technique capable of providing broadband transmission over wireless channels. Applications include wireless multimedia and Internet access as well as next-generation mobile communication systems. The advantages of multi-access over OFDM are flexibility of allocating subcarriers to users, adaptive rate and power allocation, high spectral efficiency, low receiver complexity and simple implementation by the inverse fast Fourier transform (IFFT) and FFT, see [1], [2]. OFDM can also be integrated with multiple-input multiple-output (MIMO) techniques to raise the diversity gain and increase capacity, see [2], [3].

A central problem in OFDMA is rate and power allocation of users to subcarriers. Fixed resource allocation assigns a predetermined set of subcarriers to each user. Since the scheme is fixed regardless of the current channel condition, it is far from being optimal. Subcarriers which appear in deep fade to one user may be in good condition for others. Dynamic resource allocation assigns subcarriers adaptively to users according to the current channel conditions.

In the literature, there are essentially two approaches to dynamic resource allocation. First, the so called margin adaptive (MA) objective is to minimize the overall power subject to individual data rate constraints. On the other hand, the rate adaptive (RA) problem aims at maximizing the overall transmission rate subject to power constraints. Various studies tackle the MA problem in the single-user case. In [4] a greedy bit-removing algorithm was proposed. The algorithm assigns the maximal allowable number of bits to each subcarrier initially, and then removes bit-by-bit from the subcarriers which recover the maximal transmit power. This algorithm was proved to be optimal. However, it has a fairly high complexity and is almost inapplicable for practical applications. The bit allocation in multi-user scenarios has been investigated in [5]. Fast greedy approaches are suggested in [6], [7]. Greedy algorithms are used to determine both how many and which subcarriers are assigned to each user. However, the result is often not unique and sometimes unstable.

Proportional fairness is a concept to share the medium between different user classes, where proportionality factors may be associated to billing rates. The problem of maximizing rates subject to proportional fairness and power constraints is addressed in [8], [9], [10]. Assuming a fixed subcarrier assignment the authors devise algorithms which converge to the optimal solution provided that actually all subcarriers are engaged. A standard Lagrangian and Newton-Raphson setup is used to determine optimal solutions, however, an approximation is used in the high channel-to-noise ratio case.

We suggest an alternative approach, in detail the contributions of this paper are as follows.

1. We introduce different OFDMA objectives and constraints in a single framework and directly relate power and rate optimization.
2. The single-user margin adaptive problem is solved by the easy and elegant method of directional derivatives leading to a generalized water-filling solution, interpreted in two different ways.
3. A nested algorithm is designed for solving the single-user and multi-user rate adaptive problem with proportional fairness conditions.
4. This algorithm is fully implemented and extensively tested for a class of widely used rate-power functions.

The material in this contribution is organized as follows. We start with a precise problem formulation in Section II. After introducing directional derivatives and their basic properties, the single-user OFDM optimal allocation problem is solved in Section III. Section IV deals with the optimal solution of the multi-user rate adaptive problem with proportionality constraints using iterated water-filling. In Section V the results are applied to a widely used class of rate-power functions. We also briefly report on numerical performance.

II. System Model and Problem Formulation

We consider a multi-user OFDM system with N subcarriers and K users. Each user $k \in \{1, \ldots, K\}$ has a requirement of

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R_k bits per OFDM symbol. Each subcarrier can be used by only one user at any given time.

Perfect channel state information (CSI) is assumed to be available during transmission. Let $h_{k,n}$ denote the known channel-gain of subcarrier n for user k, and $\sigma_{k,n}^2$ the according noise power. Hence, $u_{k,n} = h_{k,n}/\sigma_{k,n}^2$ is the channel-to-noise ratio (CNR). If power $p_{k,n}$ is expended on subcarrier n for the transmission to user k, then $p_{k,n}\, h_{k,n}/\sigma_{k,n}^2 = p_{k,n}\, u_{k,n}$ is the received signal-to-noise ratio (SNR).

The interrelation between power and rate is described by the nonnegative rate-power function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. $\psi\big(r_{k,n}\big)$ denotes the received power which is needed to transmit rate $r_{k,n}$ over subcarrier n to user k. It is quite natural to assume that ψ is convex with $\psi(0) = 0$ and, hence, monotone increasing. ψ depends on the maximal bit error rate (BER) that can be tolerated and on the available combinations of modulation and coding schemes.

The following constraints are used to describe the different types of problems.

$$\sum_{n=1}^{N} r_{k,n} \geq R_k, \quad k = 1, \ldots, K \tag{1}$$

$$\sum_{n=1}^{N} r_{k,n}\, r_{\ell,n} = 0, \quad k, \ell = 1, \ldots, K,\ k \neq \ell \tag{2}$$

$$r_{k,n} \geq 0, \quad k = 1, \ldots, K,\ n = 1, \ldots, N \tag{3}$$

The margin adaptive objective is to find a subcarrier assignment of minimal overall transmit power such that each user receives the required data rate. In mathematical terms this reads as

$$\min \sum_{k=1}^{K} \sum_{n=1}^{N} \frac{\psi(r_{k,n})}{u_{k,n}} \tag{4}$$

such that (1), (2) and (3) holds. The minimum overall power achieved in (4) is denoted by $p_{\mathrm{MA}}^*(\boldsymbol{R})$, where $\boldsymbol{R} = (R_1, \ldots, R_k)$ comprises the rate demands.

Introducing additional binary variables $a_{k,n} \in \{0, 1\}$ with the meaning $a_{k,n} = 1$, whenever subcarrier n is assigned to user k, and $a_{k,n} = 0$, otherwise, leads to the following equivalent formulation.

$$\min \sum_{k=1}^{K} \sum_{n=1}^{N} a_{k,n}\, \frac{\psi(r_{k,n})}{u_{k,n}} \tag{5}$$

such that (3) and the following holds.

$$\sum_{n=1}^{N} a_{k,n}\, r_{k,n} \geq R_k, \quad k = 1, \ldots, K \tag{6}$$

$$\sum_{k=1}^{K} a_{k,n} \leq 1, \quad n = 1, \ldots, N \tag{7}$$

A binary matrix $\boldsymbol{A} = (a_{k,n})_{k=1,\ldots,K,n=1,\ldots,N}$ fulfilling (7) is called subcarrier assignment. If $\boldsymbol{A}$ is fixed the margin adaptive problem reduces to solving (5) over $r_{k,n} \geq 0$ such that (6) holds. The corresponding minimum overall power is denoted by $p_{\mathrm{MA}}^*(\boldsymbol{R}, \boldsymbol{A})$.

Problems (4) and (5) represent complicated mixed continuous and combinatorial optimization problems since a joint decision on subcarrier and power allocation has to be made.

The rate adaptive objective is to find a subcarrier assignment of maximal throughput such that each user receives at least the required rate and the overall transmit power is limited by some $P > 0$. This reads as

$$\max \sum_{k=1}^{K} \sum_{n=1}^{N} r_{k,n} \tag{8}$$

such that (1), (2), (3) and

$$\sum_{k=1}^{K} \sum_{n=1}^{N} \frac{\psi(r_{k,n})}{u_{k,n}} \leq P \tag{9}$$

holds. Let $r_{\mathrm{RA}}^*(\boldsymbol{R}, P)$ denote the corresponding optimal overall rate.

Finally, we define the rate adaptive problem with proportionality constraints. In this case, rates are assumed to be proportional to a given fixed initial vector $\boldsymbol{r}^{(0)} \in \mathbb{R}_+^K$ with $\bar{r}^{(0)} = \sum_{k=1}^{K} r_k^{(0)}$. The aim is to find the maximum proportionality factor $\alpha \geq 0$ such that each user receives rate $\alpha \bar{r}^{(0)}$ and the total power budget is met. This may be formalized as

$$\max \alpha \tag{10}$$

such that (2), (3), (9) and $\sum_{n=1}^{N} r_{k,n} = \alpha r_k^{(0)}, \quad k = 1, \ldots, K$ holds. The optimal achievable rate for the proportional maximization problem is denoted by $r_{\mathrm{RA}}^*(\boldsymbol{r}^{(0)}, P)$ and the corresponding optimal factor by $\alpha_{\mathrm{RA}}^*(\boldsymbol{r}^{(0)}, P)$. For a fixed assignment $\boldsymbol{A}$ we apply analogously the notations $r_{\mathrm{RA}}^*(\boldsymbol{r}^{(0)}, P, \boldsymbol{A})$ and $\alpha_{\mathrm{RA}}^*(\boldsymbol{r}^{(0)}, P, \boldsymbol{A})$.

If there is only one user, that is $K = 1$, the problem becomes much easier. In the following we give explicit solutions to the single-user OFDM problems, which will be employed later as a building block for deriving the optimal solution of the multi-user problem (10). Directional derivatives turn out to be a powerful aid for this purpose.

III. Single-User OFDM

We start with a short description of the concept of directional derivatives and its application to the optimization of convex functions f with convex domain $\mathcal{C}$. Let $x_0, x \in \mathcal{C}$. The directional derivative of f at x_0 in the direction of x is defined as

$$\begin{aligned} Df(x_0, x) &= \lim_{\beta \to 0^+} \frac{1}{\beta} \Big[f\big((1-\beta)x_0 + \beta x\big) - f(x_0) \Big] \\ &= \frac{d}{d\beta} f\big((1-\beta)x_0 + \beta x\big)\Big|_{\beta=0^+}, \end{aligned} \tag{11}$$

see, e.g., [11]. Since f is convex, $\big(f((1-\beta)x_0 + \beta x) - f(x_0)\big)/\beta$ is monotone decreasing with decreasing $\beta \geq 0$, and the directional derivative always exists.

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If $\mathcal{C}$ is a subset of a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, it is well known that

$$Df(x_0,x) = \langle \nabla f(x_0), x - x_0\rangle, \tag{12}$$

whenever ∇f, the derivative of f, exists.

Optimal points are characterized by directional derivatives as follows, for a proof see [11].

Proposition 1: Let $\mathcal{C}$ be a convex set and $f : \mathcal{C} \to \mathbb{R}$ a convex function. Then the minimum of f is attained at x^* if and only if $Df(x^*,x) \geq 0$ for all $x \in \mathcal{C}$.

We apply this principle to solving the single-user OFDM margin adaptive problem with N subcarriers. Let

$$\mathcal{R} = \left\{ \boldsymbol{r} = (r_1,\dots,r_N) \mid r_i \geq 0, \textstyle\sum_{i=1}^N r_i = R \right\}$$

denote the admissible rate region. The power allocation problem (4) then reads as

$$\min_{\boldsymbol{r}\in\mathcal{R}} \sum_{i=1}^N \frac{\psi(r_i)}{u_i}. \tag{13}$$

Note, that the optimal solution is attained at the boundary of constraint (1) as the rate-power function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous (due to convexity). Hence, the function $f(\boldsymbol{r}) = \sum_{i=1}^N \psi(r_i)/u_i$ is convex as well. The directional derivative of f exists, and for differentiable ψ it is easily determined as

$$Df(\hat{\boldsymbol{r}},\boldsymbol{r}) = \sum_{i=1}^N \frac{\psi'(\hat{r}_i)}{u_i}\,(r_i - \hat{r}_i). \tag{14}$$

By Proposition 1, some rate allocation $\hat{\boldsymbol{r}} \in \mathcal{R}$ is optimal if and only if $Df(\hat{\boldsymbol{r}},\boldsymbol{r}) \geq 0$ for all $\boldsymbol{r} \in \mathcal{R}$. Applying this condition to (14) yields that $\hat{\boldsymbol{r}} \in \mathcal{R}$ is optimal if and only if

$$\frac{\psi'(\hat{r}_i)}{u_i} = \lambda \text{ for all } i \text{ with } \hat{r}_i > 0 \tag{15}$$

for some constant $\lambda \geq 0$. Equation (15) is equivalent to the following

$$\begin{aligned} \hat{r}_i &= \begin{cases} \psi'^{-1}(\lambda u_i), & \text{if } \lambda u_i > \psi'(0), \\ 0, & \text{otherwise,} \end{cases} \\ \lambda &\text{ such that } \sum_{i=1}^N \hat{r}_i = R. \end{aligned} \tag{16}$$

$\psi'(0)$ is understood as the right sided derivative of the convex rate-power function ψ at 0, which exists. Hence, optimality is also characterized by condition (16).

The solution of the single-user rate adaptive problem (8) is closely related to the optimal solution of the single-user margin adaptive problem (4). Note that in the single-user case $R \geq 0$ is a one-dimensional parameter.

Proposition 2: The optimal solutions of the single-user rate and margin adaptive problem satisfy

$$p^*_{\text{MA}}(r^*_{\text{RA}}(R,P)) = P.$$

Furthermore, for all $R' \geq R$ it holds that

$$r^*_{\text{RA}}(R, p^*_{\text{MA}}(R')) = R'. \tag{17}$$

The above is easy to prove, bearing in mind that the optimal solutions are obtained at the boundary of the inequality constraints (1) and (9), respectively. Using Proposition 2 and solution (16) of the margin adaptive problem a solution of the rate adaptive problem is obtained as

$$r^*_{\text{RA}}(R,P) = \max\{R' \geq R \mid p^*_{\text{MA}}(R') \leq P\}.$$

This result is carried over to the multi-user problem in the next section.

IV. Iterated Water-Filling

We consider the multi-user rate adaptive problem with proportionality constraints for the case of some fixed predetermined subcarrier allocation $\boldsymbol{A}$. Accordingly, additional constraints

$$r_{k,n} = 0, \text{ whenever } a_{k,n} = 0, \tag{18}$$

$k = 1,\dots,K,\ n = 1,\dots,N$, are introduced. The additional $\boldsymbol{A}$ in the definitions of p^*_{MA}, r^*_{RA} and α^*_{RA} indicates that the corresponding problems are solved subject to a given assignment $\boldsymbol{A}$ by constraints (18).

The following Proposition is analogous to Proposition 2 in the single-user case.

Proposition 3: For the multi-user OFDM problems with fixed assignment $\boldsymbol{A}$ it holds that

$$p^*_{\text{MA}}\big(\alpha^*_{\text{RA}}(\boldsymbol{r}^{(0)},P,\boldsymbol{A})\,\boldsymbol{r}^{(0)},\boldsymbol{A}\big) = P.$$

Furthermore, for all $\alpha' \geq \alpha$ it holds that

$$r^*_{\text{RA}}\big(\alpha\,\boldsymbol{r}^{(0)}, p^*_{\text{MA}}(\alpha'\boldsymbol{r}^{(0)},\boldsymbol{A}),\boldsymbol{A}\big) = \alpha'\bar{r}^{(0)}.$$

A direct consequence of Proposition 3 is the following.

Proposition 4: The optimal solution of the rate and margin adaptive problem subject to proportionality constraints and a fixed assignment $\boldsymbol{A}$ satisfy

$$\begin{aligned} &r^*_{\text{RA}}(\boldsymbol{r}^{(0)},P,\boldsymbol{A}) \\ &\quad = \bar{r}^{(0)} \max\{\alpha \in \mathbb{R}_+ \mid p^*_{\text{MA}}(\alpha\boldsymbol{r}^{(0)},\boldsymbol{A}) \leq P\}. \end{aligned}$$

Hence, a solution to the proportional rate adaptive problem may be determined by a nested algorithm where in the outer loop α is increased and in the inner loop $p^*_{\text{MA}}(\alpha\boldsymbol{r}^{(0)},\boldsymbol{A})$ is computed until the maximum power P is reached.

Since the assignment of subcarriers is fixed by $\boldsymbol{A}$, determining $p^*_{\text{MA}}(\alpha\boldsymbol{r}^{(0)},\boldsymbol{A})$ decomposes into K separate subproblems as

$$p^*_{\text{MA}}(\alpha\boldsymbol{r}^{(0)},\boldsymbol{A}) = \sum_{k=1}^K p^*_{\text{MA}}(\alpha r_k^{(0)}). \tag{19}$$

Each $p^*_{\text{MA}}(\alpha r_k^{(0)})$ means solving a single-user margin adaptive problem by generalized water-filling (16), which can be done efficiently. Details of how to implement the necessary steps in an efficient way are given in the next chapter for a general class of rate-power functions.

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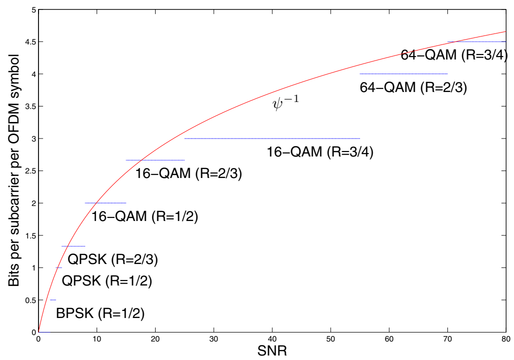


Fig. 1. Power-Rate Function, cf. [7]

V. A Concrete Class of Rate-Power Functions

As an approximation to actually implementable rate-power functions we use $\psi(r) = a(2^r - 1)$, $r \geq 0$, depicted in Fig. 1. It is widely used in the literature (e.g., [12]) and well motivated by achievable rates using PSK and QAM modulation schemes, see [13]. The function, its inverse, and the respective derivatives are given by

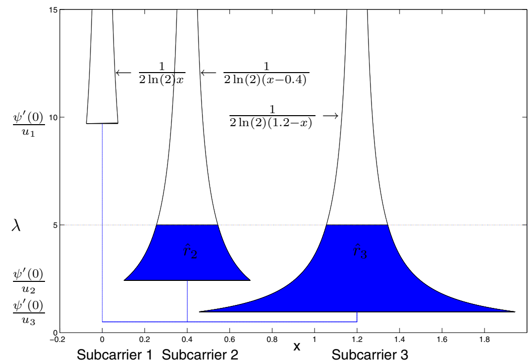
$$\begin{aligned} \psi(r) &= a\big(2^r - 1\big), && r \geq 0, \\ \psi^{-1}(p) &= \log_2\left(1 + \frac{p}{a}\right), && p \geq 0, \\ \psi'(r) &= a(\ln 2)2^r, && r \geq 0, \\ \psi'^{-1}(y) &= \log_2\left(\frac{y}{a \ln 2}\right), && y \geq a\ln(2). \end{aligned} \tag{20}$$

Parameter $a > 0$ depends on the maximal acceptable BER and the available combinations of modulation and coding schemes.

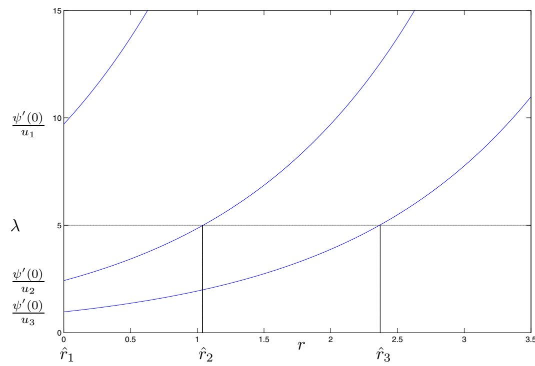
Fig. 2 shows the optimal rate allocation for a single-user margin adaptive problem with three subcarriers and parameters $u_1 = 0.05$, $u_2 = 0.2$, $u_3 = 0.5$, and $a = 0.7$. The value $\lambda = 5$ yields rates $r_1 = 0$, $r_2 = 1.05$, $r_3 = 2.4$ corresponding to an overall rate $R = 3.45$. Fig. 2 also demonstrates the generalized water-filling principle. If $\psi'(r)$ were a linear function, $\psi'(r) = \psi_o + r$, say, then the values $\hat{r}_i$ would represent the water-pouring height on top of $\frac{\psi_0}{u_i}$ to achieve constant water level λ. Function ψ' distorts the amount of water filled onto $\frac{\psi'(0)}{u_i}$. In Fig. 2(a), the shaded area corresponds to the amount of water filled into the system of inverted goblets connected by thin tubes. In general, the shape of the bins is determined by the function

$$h(x, u) = \frac{u}{2\psi''(\psi'^{-1}(xu))}, \quad x \geq \frac{a\ln(2)}{u}.$$

In the following we describe how to determine the single-user water-filling solution (16) for rate-power functions of type (20). Let $U^m = \{i_1, \ldots, i_m\}$ denote the set of indices corresponding to the m largest CNRs $u_{i_1} \geq \cdots \geq u_{i_m}$ and $\boldsymbol{u}^m = (u_{i_1}, \ldots, u_{i_m})$. gMean$(\boldsymbol{u}^m)$ and hMean$(\boldsymbol{u}^m)$ denote the geometric and harmonic mean, respectively, of the components of $\boldsymbol{u}^m$. Let m^* be the largest $m \leq N$ such that



(a) Generalized Water-Filling



(b) Rate Allocation

Fig. 2. An optimal rate allocation for $\psi'(r) = 0.7(\ln 2)2^r$, three subcarriers with $u_1 = 0.05$, $u_2 = 0.2$, $u_3 = 0.5$. $\lambda = 5$ yields the rates $r_1 = 0$, $r_2 = 1.05$, $r_3 = 2.4$ corresponding to a total rate $R = 3.45$.

$\log_2\left(\frac{u_{i_m}}{\text{gMean}(\boldsymbol{u}^m)}2^{R/m}\right)$ is positive. The optimal solution (16) is then given as

$$\begin{aligned} \lambda^* &= \frac{a\ln(2)}{\text{gMean}(\boldsymbol{u}^{m^*})}2^{R/m^*}, \\ r_i^* &= \begin{cases} \log_2\left(\frac{u_i}{\text{gMean}(\boldsymbol{u}^{m^*})}2^{R/m^*}\right), & \text{if } i \in U^{m^*} \\ 0, & \text{otherwise} \end{cases}, \end{aligned} \tag{21}$$

with optimal value

$$\begin{aligned} p^*_{\text{MA}}(R) &= \sum_{i \in U^{m^*}} \frac{\psi(r_i^*)}{u_i} \\ &= am^*\left[\frac{1}{\text{gMean}(\boldsymbol{u}^{m^*})}2^{\frac{R}{m^*}} - \frac{1}{\text{hMean}(\boldsymbol{u}^{m^*})}\right]. \end{aligned} \tag{22}$$

Note that parameter a has no influence on the optimal rates. However, the optimal power is proportional to that value. From Proposition 2 the optimal rate for the single-user rate adaptive problem is derived as

$$\begin{aligned} &r^*_{\text{RA}}(R, P) = m^*_{\text{RA}} \cdot \\ &\quad \log_2\left(gMean(\boldsymbol{u}^{m^*_{\text{RA}}})\left[\frac{P}{am^*_{\text{RA}}} + \frac{1}{hMean(\boldsymbol{u}^{m^*_{\text{RA}}})}\right]\right) \end{aligned} \tag{23}$$

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Algorithm 1 ITERATEDWATERFILLING(Power P)

$\alpha_{min} \leftarrow$GETMINIMALALPHA() {with $\gamma_k = g_k(1)$, cf. (25)}
$\alpha_{max} \leftarrow$GETMAXIMALALPHA()
$p_{min} \leftarrow$GETPOWER(α_{min}) {cf. (24)}
$p_{max} \leftarrow$GETPOWER(α_{max})
while $\left(1 - \frac{p_{min}}{P} > \epsilon\right)$ **do**
  $\alpha_{min} \leftarrow \alpha_{min} + (\alpha_{max} - \alpha_{min}) \frac{P - p_{min}}{p_{max} - p_{min}}$
  $p_{min} \leftarrow$GETPOWER(α_{min})
end while

The number of used channels for the margin and rate adaptive objective may differ. Therefore, m^*_{RA} has to be chosen as the largest m such that the smallest rate $\log_2\left(u_{i_m}\left[\frac{P}{am} + \frac{1}{\mathrm{hMean}(\boldsymbol{u}^m)}\right]\right)$ is positive. Combining decomposition (19) and equation (22) for the multi-user case gives

$$\begin{aligned} &p^*_{\mathrm{MA}}(\alpha \boldsymbol{r}^{(0)}, \boldsymbol{A}) \\ &= \sum_{k=1}^{K} a m^*_k \left[\frac{1}{\mathrm{gMean}(\boldsymbol{u}_k^{m^*_k})} 2^{\frac{\alpha r_k^{(0)}}{m^*_k}} - \frac{1}{\mathrm{hMean}(\boldsymbol{u}_k^{m^*_k})}\right]. \end{aligned} \tag{24}$$

Formula (24) can be efficiently computed such that the nested algorithm derived from Proposition 4 is applicable. The outer iteration is considered in the following. We first give a lower and upper bound which is of great help in the numerical computations.

Proposition 5: For a multi-user rate adaptive problem with proportionality constraints

$$\min \frac{r^*_{\mathrm{RA}}(\gamma_k P)}{r_k^{(0)}} \le \alpha^*_{\mathrm{RA}} \le \max \frac{r^*_{\mathrm{RA}}(\gamma_k P)}{r_k^{(0)}}$$

holds for all $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_K) \in \mathbb{R}^K_+$ with $\sum_{k=1}^K \gamma_k = 1$.

The algorithm ITERATEDWATERFILLING(Power P) numerically computes the optimal parameter α^*_{RA} of a given multi-user rate adaptive problem with proportionality constraints. It stops if the relative error of the current power p_{min} is less than ϵ. First, α^*_{RA} is limited by the bounds of Proposition 5. Excellent results were obtained by the intuitively motivated choice $\gamma_k = g_k(1)$ with

$$g_k(x) = \frac{p^*_{\mathrm{MA}}(x r_k^{(0)})}{\sum_{l=1}^K p^*_{\mathrm{MA}}(x r_l^{(0)})}. \tag{25}$$

In the algorithm, the corresponding powers p_{min} and p_{max} are computed according to (24). The overall transmit power is convex in the rates because it is a linear combination of convex functions ψ. Consequently, its inverse is concave. The rate is proportional to the water level α, such that the function representing the water level given a power is concave. This ensures that the updates of α_{min} stay below the optimal level α^*_{RA} within each loop. The updates converge strictly monotonic increasing to α^*_{RA} corresponding to the desired power P. Fig. 3 illustrates the first three steps of the algorithm. The complexity of the algorithm is $O(K)$, as functions involved have complexity $O(K)$.

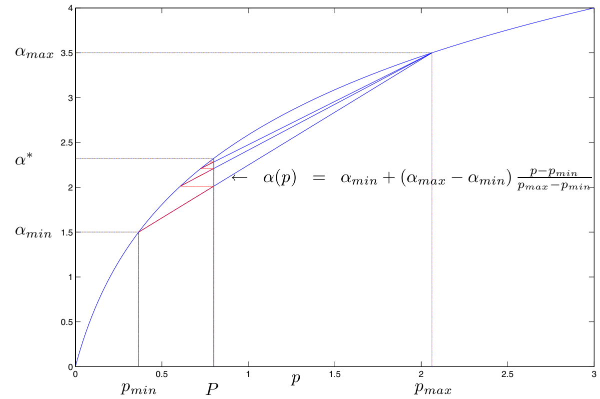

Fig. 3. Visualization of ITERATEDWATERFILLING(P)

Our algorithm cannot be directly compared to [10], since in that paper an approximation is used leading to a violation of the proportionality constraints. However for benchmarking purposes we have implemented a standard Newton-Raphson method for computing α^*_{RA}. Algorithm 1 is slightly superior. In summary, the nested algorithm converges extremely fast to the optimal proportional rate allocation, normally using less than four iterations to achieve error bound $\epsilon = 10^{-4}$.

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