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Generalised multi-receiver radio network: Capacity and asymptotic stability of power control through Banach's fixed-point theorem

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***Abstract*—We introduce a model of a generalised multi-receiver radio network with quality-of-service (QoS) constraints. There are two key functions: (1) Ni is non-decreasing and homogeneous and gives i's QoS as function of its carrier-to-interference ratios at each of K receivers, (2) nu_ik is a semi-norm that gives the interference experienced by transmitter i at receiver k as function of the power vector. We utilise "norm" concepts and Banach's well-known fixed-point theorem to characterise the conditions under which a QoS vector is feasible, and the corresponding power-adjustment process converges. The critical power levels equal a_i/h_i where a_i is the QoS target, and h_i is the 'average' channel gain. hi=Ni(h_i1, ..., h_i,K) where h_ik is the channel gain from transmitter i to receiver k. If the interference experienced by each transmitter i at each receiver k is less than 1 when each power is set to the critical level (i.e., nu_ik(a_1/h_1, a_2/h_2,..., a_N/h_N)<1), then the QoS targets are feasible. Applications of our general result yields simple feasibility formulae for 3 scenarios from Yates (JSAC, 13(7):1341-1348, 1995): (i) fixed base-station assignment, (ii) macro-diversity and, (iii) multiple-connection reception (terminal must maintain acceptable QoS at several receivers), which includes the minimum power assignment as a special case. The 3 formulae have the simple form: largest weighted sum of N-1 QoS targets less than one, where the weights are relative channel gains.**

I. Introduction

In certain wireless communication scenarios, such as in the terminal-to-base-station link of a CDMA cellular network, a terminal chooses its transmission power by following an adjustment rule which depends on the power levels of the other terminals. Thus, terminal i may choose a power p_i given by a function $g_i(\mathbf{p}_{-i})$, where $\mathbf{p}_{-i}$ denotes the vector of the power levels of each of the other terminals. Generally, $g_i(\mathbf{p}_{-i})$ is such that terminal i's quality of service (QoS) — usually assessed through the carrier-to-interference ratio (CIR) — is kept above a certain level.

A fundamental question is: under which conditions does the power adjustment process converge, in the sense that it reaches a power vector $\mathbf{p}^*$ such that for each i, $p_i^* = g_i(\mathbf{p}^*{}_{-i})$ (that is, each adjustment rule indicates no further change, given the level of power of the other terminals)? Since it may be difficult for the system to get each terminal to start the iterations at a specified power level, it is also interesting to know whether the adjustment process is asymptotically stable, in the sense that it always converges to the same power vector, $\mathbf{p}^*$, regardless of the initial power levels. Finally, convergence should depend on the QoS targets. It is interesting to characterise this dependence, preferably in closed form, so that, for instance, the system can determine whether a terminal wishing service at certain QoS level can be admitted, without breaking QoS commitments made to "incumbent" terminals. For example, for the specific case of a CDMA wireless communication system in which base stations "cooperate" (macro-diversity), [1] shows under certain approximation that if the sum of the desired CIR's is less than the number of receivers these quality-of-service targets are feasible, and the pertinent adjustment process always converges to a unique power vector. More recently, [2] presents a still simple but more sophisticated condition that — through a dependence on *relative* channel gains — sensibly adjusts itself to special channel states, such as when terminals are in range of only a subset of the receivers. The set of the feasible QoS vectors is associated with the "capacity region" of the system (see [3], [4] for relevant discussion).

A relevant strand of the scientific literature seeks conditions for the convergence of a process in which terminals take turns, each "greedily" choosing a power level in order to achieve its desired QoS while taking the other power levels as fixed. This literature focuses on an abstract model in which all details of the physical system are "hidden" inside the power adjustment functions, about which *all that is known* is that they have certain simple properties. This approach is important because its results apply to all practical systems that can be shown to satisfy the assumed properties. In particular, [5] shows that if the "interference functions" are non-negative, non-decreasing, and — in certain sense — (sub)homogeneous, greedy power adjustment converges to a unique vector, *provided* that the underlying QoS targets are *feasible*. Related work includes [6], which extends [7] to a "canonical class" of algorithms that can handle discrete power levels; [8] which focuses on opportunistic power adjustment for delay-tolerant traffic; and [4], which considers interference functions whose properties

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are similar but non-identical to those in [5]. However, none of these works provides explicit feasibility conditions for the corresponding family of functions.

The present work — which can be viewed as a direct generalisation of the model and analysis of [2] — includes as special cases a plethora of models, such as all those discussed in [5]. The most significant difference between the related work [9] and the present one lies in their degree of "abstraction". Within the abstract model mentioned above, [9] provides a feasibility condition for a broad class of power adjustment functions. By contrast, details such as channel gains, QoS constraints, and the number of receivers are explicitly considered herein. The abstract "high-level view" of [5] and its descendants is evidently the most general; however the more concrete model introduced here may yield insights and opportunities otherwise unavailable. Thus, both approaches are complementary. On the other hand, the present model has the same level of abstraction as [10]'s, but [10] focuses on network simplification, while assuming less about the interference function (they need *not* be sub-additive), and utilising simpler analytical tools.

In our model, associated with each of N terminals there are 2 key functions: (i) $\nu_{i,k}$, a monotonic (semi)norm which gives the interference experienced by terminal i at receiver k as a function of the power vector, and (ii) $\mathcal{N}_i$, a non-decreasing, homogeneous function that takes as input i's carrier-to-interference ratios at the K receivers and yields i's QoS. The main contribution herein is a simple result that can tell us for a generalised multi-receiver network whether a set of QoS targets is feasible, which, in particular, is useful to prevent — through admission control — the waste of valuable resources: If each $\nu_{i,k}$ satisfies $\nu_{i,k}(\alpha_1/h_1, \alpha_2/h_2, \cdots, \alpha_N/h_N) < 1$, where α_i is the QoS target, and $h_i := \mathcal{N}_i(h_{i,1}, \cdots, h_{i,K})$ (h_i is the 'representative' channel gain) then the QoS targets are feasible. The application of this result yields simple "capacity" formulae — similar to those given by [1],[2] for macro-diversity — for other scenarios from [5], such as the challenging multiple connection reception — which includes the minimum power assignment as a special case — for which, to our knowledge, no previous such formula existed.

Below, after describing the basic system model and the generalised QoS constraint, our results are derived on the basis of "norm" concepts, and Banach's fixed-point theorem [11], [12]. A norm is a well-understood and intuitive generalisation of the "length" of a vector, which has been fruitfully applied in many contexts, including beam-forming [13]. Two appendices provide the essential mathematical background, and some technical results from the literature. A third appendix gives proofs directly linked to our analysis.

II. Quality of Service in a Generalised Multi-Receiver Radio Network

A. QoS model

Consider a system with N transmitters and K receivers, such as in the reverse-link of a cellular system. A generalised quality-of-service (QoS) index for terminal i is defined as:

$$\mathcal{N}_i(\rho_{i,1}, \cdots, \rho_{i,K}) \geq \alpha_i \tag{1}$$

where $\mathcal{N}_i : \Re_+^K \to \Re_+$ is a non-negative function that satisfies positive homogeneity (i.e., $\mathcal{N}_i(\lambda \mathbf{x}) = \lambda \mathcal{N}_i(\mathbf{x})$ for all $\mathbf{x} \in \Re_+^K$ and $\lambda \in \Re_+$), and

$$\rho_{i,k} := \frac{P_i h_{i,k}}{Y_{i,k} + \sigma_k} \tag{2}$$

where

- P_i is the transmission power level of terminal i,
- $h_{i,k}$ is the channel gain in the signal from terminal i arriving at receiver k,
- σ_k is the (average) power of the additive random noise at receiver k, and
- $Y_{i,k} := \nu_{i,k}(\mathbf{P})$ where $\mathbf{P}$ is the vector of the transmission power levels of each terminal, and $\nu_{i,k}$ denotes some (semi-)norm (Definition A.1). Thus, $\nu_{i,k}$ is a general measure of the "size" of the interfering power experienced by transmitter i at receiver k.

Below, we recognise and utilise the vectors:

$$\mathbf{Y}_i := (Y_{i,1}, \cdots, Y_{i,K}) \tag{3}$$

$$\mathbf{H_i} := (h_{i,1}, \cdots, h_{i,K}) \tag{4}$$

$$\sigma := (\sigma_1, \cdots, \sigma_K) \tag{5}$$

$$\rho_i = (\rho_{i,1}, \cdots, \rho_{i,K}) \tag{6}$$

When not otherwise indicated, homogeneity is of degree one.

The preceding model encompasses numerous interesting scenarios, including, as discussed below, all the examples cited by [5].

B. Examples

Below, we write the absolute value operator over some quantities that we know are non-negative, simply to emphasise that certain functions satisfy Definition A.1.

For all the examples below,

$$\nu_{i,k}(\mathbf{P}) := \sum_{\substack{n=1 \\ n \neq i}}^{N} \left| P_n h_{n,k} \right| \tag{7}$$

By Lemma C.1, (7) defines a norm on $\Re^{N-1}$ and a seminorm on $\Re^N$.

1) Macro-diversity (MD): b

Under macro-diversity, the relevant QoS constraint is [1] :

$$\frac{P_i h_{i,1}}{Y_{i,1} + \sigma_1} + \cdots + \frac{P_i h_{i,K}}{Y_{i,K} + \sigma_K} \geq \alpha_i \tag{8}$$

Thus, $\mathcal{N}_i^{\mathrm{MD}}(x_1, \cdots, x_K) = |x_1| + \cdots + |x_K|$; this is the Hölder 1-norm (Definition A.3).

2) Multiple-connection reception (MC): Under MC, user i must maintain an acceptable SIR α_i at d_i distinct receivers. The system "assigns" i to the d_i "best" receivers. For $\mathbf{x} \in \Re^N$ and $M \in \mathbb{N}$, $M < N$, let $\max(x;M)$ denote the Mth largest *absolute value* of the components of $\mathbf{x}$. The QoS requirements of i can be written as [5]:

$$\max\left(\left(\frac{P_i h_{i,1}}{Y_{i,1}+\sigma_1},\cdots,\frac{P_i h_{i,K}}{Y_{i,K}+\sigma_K}\right);d_i\right) \geq \alpha_i \tag{9}$$

Thus, $\mathcal{N}_i^{\mathrm{MC}}(x_1,\cdots,x_K) = \max((x_1,\cdots,x_K);d_i)$. By Lemma C.3 and Remark C.2, $\mathcal{N}_i^{\mathrm{MC}}$ is a semi-norm, and hence has the desired properties.

With $d_i = 1$, the MC scheme becomes the minimum power assignment (MPA) [5].

3) Fixed base-station assignment (FA): Let k_i denote the index of the base station receiver to which i has been assigned. The corresponding QoS constraint can be written as

$$\rho_{i,k_i} = \frac{P_i h_{i,k_i}}{Y_{i,k_i}+\sigma_{k_i}} \geq \alpha_i \tag{10}$$

Thus, $\mathcal{N}_i^{\mathrm{FA}}(x_1,\cdots,x_K) = |x_{k_i}|$. This function simply "picks" the component of ρ that corresponds to k_i. With $e^i \in \Re^K$ denoting the unit vector whose only non-zero component is the k_ith component, then $\mathcal{N}_i^{\mathrm{FA}}(x_1,\cdots,x_K)$ can be written as a dot product: $\sum_{k=1}^K \left|e_k^i x_k\right|$. $\mathcal{N}_i^{\mathrm{FA}}$ is evidently homogeneous in $\mathbf{x}$; in fact, it can be directly verified that it satisfies Definition A.1.

III. Capacity as a fixed-point problem

A. The capacity question

Conditions are sought under which a given N-vector of positive numbers, $\alpha := (\alpha_1,\cdots,\alpha_N)$, is such that there exists another N-vector $\mathbf{P} = (P_1,\cdots,P_N)$ that satisfies appropriate constraints, as well as condition (1) for each i. When such solution exists, the vector of QoS targets α is said to be in the "capacity region" of the system.

B. Banach's fixed-point approach

One can envision a process in which terminals take turns adjusting transmission power, and each terminal chooses the power level that achieves its desired QoS under the present level of interference. Of course, when a terminal changes its power, it also alters the level of interference experienced by the others. Thus, a terminal which may have previously adjusted its power may need to do so again. If this adjustment process "converges" to a feasible power vector, in the sense that at the present power levels, no terminal needs further power adjustment to achieve its QoS, evidently, those QoS targets are feasible.

If we denote as $\mathbf{T}(\mathbf{p})$ the transformation that produces a new power vector as a function of the present one, when each terminal adjusts its power as described above, then we need to characterise the conditions under which there is a vector $\mathbf{p}^*$ such that $\mathbf{p}^* = \mathbf{T}(\mathbf{p}^*)$, that is, $\mathbf{p}^*$ is a fixed point of $\mathbf{T}$.

Theorem B.1 holds that, if $\mathbf{T}$ is a *contraction* (Definition B.1), then $\mathbf{T}$ has a *unique* fixed-point, and it can be found via successive approximation (Definition B.2).

C. Normalised adjustment

1) Replacing each component of a vector with its norm: Constraint (1) will often lead to a power adjustment process with which it is very difficult to work. However, one can obtain conservative feasibility results by replacing each $Y_{i,k}(\mathbf{P})$ with

$$\|\mathbf{Y}_i\|_\infty \equiv \max_k\{Y_{i,k}\} \tag{11}$$

and each σ_k with

$$\|\sigma\|_\infty \equiv \max_k\{\sigma_k\} \tag{12}$$

Now constraint (1) leads to:

$$\mathcal{N}_i\left(\frac{P_i h_{i,1}}{\|\mathbf{Y_i}\|_\infty+\|\sigma\|_\infty},\cdots,\frac{P_i h_{i,K}}{\|\mathbf{Y_i}\|_\infty+\|\sigma\|_\infty}\right) \geq \alpha_i \tag{13}$$

Since $\mathcal{N}_i$is homogeneous, condition (13) can be written as:

$$\frac{P_i h_i}{\|\mathbf{Y_i}\|_\infty+\|\sigma\|_\infty} \geq \alpha_i \tag{14}$$

where

$$h_i := \mathcal{N}_i(\mathbf{H_i}) \equiv \mathcal{N}_i(h_{i,1},\cdots,h_{i,K}) \tag{15}$$

$\mathcal{N}_i$ can be any member of the class of positively homogeneous functions, which includes all (semi-)norms. If $\mathcal{N}_i$ is (interpreted as) a norm, then h_i is the "size" (norm) of the vector of channel gains, $\mathbf{H_i}$ ('average' channel gain). Thus, one can interpret the left side of (14) as the ratio of the 'average' received power from i to the sum of the 'average' interference and the 'average' noise power; i.e., a (generalised) carrier-to-noise-plus-interference ratio.

2) Adjustment process: With (14) taken as equality, and the super-index denoting — when not omitted — the iteration step, the power adjustment process can be written as $P_i^{t+1} = f_i^t(\mathbf{P}) + c_i$ where,

$$f_i(\mathbf{P}) := \frac{\alpha_i}{h_i}\|\mathbf{Y_i}\|_\infty \equiv \frac{\alpha_i}{h_i}\|(\nu_{i,1}(\mathbf{P}),\cdots,\nu_{i,K}(\mathbf{P}))\|_\infty \tag{16}$$

and

$$c_i := \frac{\alpha_i}{h_i}\|\sigma\|_\infty \tag{17}$$

(14) suggests the change of variable:

$$q_i := \frac{h_i P_i}{\alpha_i} \tag{18}$$

Then, the adjustment can be re-written as $q_i^{t+1} = f_i^t(\mathbf{q}) + \hat{c}_i$ where,

$$f_i(\mathbf{q}) := \|\mathbf{Y_i}\|_\infty \equiv \|(\nu_{i,1}(\mathbf{q}),\cdots,\nu_{i,K}(\mathbf{q}))\|_\infty \tag{19}$$

and

$$\hat{c}_i := \|\sigma\|_\infty \tag{20}$$

D. Key technical result

To answer the capacity question, we shall identify the conditions under which the transformation that returns a new power vector as a function of the present one is a contraction (Definition B.1). Under those conditions, Theorem B.1 holds that $\mathbf{T}$ has a fixed point, and therefore the corresponding QoS vector is feasible. We shall characterise the desired conditions, by invoking Lemma C.5.

We shall focus on the scaled power vector, $\mathbf{q}$. Below, $\vec{1}_M$ denotes the element of $\Re^M$ with each component equal to 1.

Theorem III.1: If each $\nu_{i,k}$ satisfies (i) Definition A.1, (ii) $\nu_{i,k}(\mathbf{x}) \leq \nu_{i,k}\left(\|\mathbf{x}\|_\infty \vec{1}_N\right) \quad \forall \mathbf{x} \in \Re^N$ (monotonicity), and (iii) $\nu_{i,k}(\vec{1}_N) < 1$, then the transformation $\mathbf{T}$ defined by $T_i(\mathbf{q}) = \|(\nu_{i,1}(\mathbf{q}), \cdots, \nu_{i,K}(\mathbf{q}))\|_\infty + \|\sigma\|_\infty$ for $\mathbf{q} \in \Re^N$, $i = 1 \ldots N$ satisfies Definition B.1.

Proof:

Let f_i be defined by $f_i(\mathbf{q}) := \|(\nu_{i,1}(\mathbf{q}), \cdots, \nu_{i,K}(\mathbf{q}))\|_\infty$.

By definition of f_i, $\nu_{i,k}(\mathbf{x}) \leq \nu_{i,k}\left(\|\mathbf{x}\|_\infty \vec{1}_N\right) \quad \forall \mathbf{x} \in \Re^N \implies f_i(\mathbf{x}) \leq f_i\left(\|\mathbf{x}\|_\infty \vec{1}_N\right) \quad \forall \mathbf{x} \in \Re^N$

By Lemma C.4, if each $\nu_{i,k}$ satisfies Definition A.1, so does f_i.

$\nu_{i,k}(\vec{1}_N) < 1 \, \forall k \implies \max\left(\nu_{i,1}(\vec{1}_N), \cdots, \nu_{i,K}(\vec{1}_N)\right) \equiv f_i(\vec{1}_N) < 1 \, \forall i$

Therefore, Lemma C.5 can be applied, and the desired result follows. ■

E. Capacity implications

The key feasibility condition arising from Theorem III.1 is that if each $\nu_{i,k}(\mathbf{q}) < 1$ when each $q_i = h_i P_i / \alpha_i = 1$ (i.e., $P_i = \alpha_i / h_i$), then each α_i is feasible. This result is applied below to the specific scenarios of subsection II-B.

1) General form: With the new coordinates, $P_n h_{n,k} \equiv q_n \alpha_n h_{n,k} / h_n \equiv q_n \alpha_n g_{n,k}$, where :

$$g_{i,k} := \frac{h_{i,k}}{h_i} \tag{21}$$

Corresponding to equation (7) we now have

$$\nu_{i,k}(\mathbf{q}) := \sum_{\substack{n=1 \\ n \neq i}}^{N} \left| q_n \alpha_n g_{n,k} \right| \tag{22}$$

Now, the feasibility condition of Theorem III.1 leads to

$$\sum_{\substack{n=1 \\ n \neq i}}^{N} \alpha_n g_{n,k} < 1 \quad \forall i, k \tag{23}$$

Thus, the greatest weighted sum of $N-1$ carrier-to-interference ratios must be less than 1, in order for α to lie in the capacity region of the system. The weights are relative channel gains. At most NK such simple sums need to be checked before an admission decision.

Condition (23) applies to all the cases discussed in subsection II-B, because the expression for $\nu_{i,k}$ given by equation (7) works for all cases. However, h_i (and, hence, $g_{i,k}$) is different for each case.

2) Fixed assignment: $h_i = h_{i,k_i}$ where k_i is the index of the base station receiver to which i has been assigned. Thus, $g_{i,k} = h_{i,k} / h_{i,k_i}$. Then, condition (23) becomes:

$$\sum_{\substack{n=1 \\ n \neq i}}^{N} \frac{h_{n,k}}{h_{n,k_n}} \alpha_n < 1 \quad \forall i, k \tag{24}$$

Notice that if $K = 1$, then $g_{i,1} = 1$ for each i, and the condition reduces to $\sum_{\substack{n=1 \\ n \neq i}}^{N} \alpha_n < 1 \quad \forall i$. $\sum_{\substack{n=1 \\ n \neq i}}^{N} \alpha_n$ adds all α_i except one; such sum is, evidently, largest when it leaves out the smallest α_i. If one re-labels the terminals so that $\alpha_N \leq \alpha_i \quad \forall i$, the condition takes the simpler form $\sum_{n=1}^{N-1} \alpha_n < 1$.

3) Macro-diversity: $h_i = \sum_k h_{i,k}$ and $g_{i,k} = h_{i,k} / \sum_k h_{i,k}$. Condition (23) specialises as:

$$\sum_{\substack{n=1 \\ n \neq i}}^{N} \frac{h_{n,k}}{\sum_k h_{n,k}} \alpha_n < 1 \quad \forall i, k \tag{25}$$

This condition and its advantages over that provided by [1] are discussed extensively in [2]. In particular, it is closest to [1]'s in the special case in which each terminal is "equidistant" from each receiver; that is, for each i, $h_{i,k} \approx h_{i,l} \, \forall k, l$ (for example, the terminals may be distributed along a line that is perpendicular to the axis between the 2 symmetrically placed receivers). In this case, each $g_{n,k} = 1/K$, and condition (25) reduces to $\sum_{\substack{n=1 \\ n \neq i}}^{N} \alpha_n < K$ for each i (which is consistent with condition (24), for $K = 1$). By comparison, [1] gives the condition $\sum_{n=1}^{N} \alpha_n < K$ for all cases.

4) Multiple-connection reception: $h_i = \max((h_{i,1}, \cdots, h_{i,K}); d_i)$ with d_i the number of receivers with which i must maintain acceptable quality of service. E.g. if $d_i = 3 < K$, then h_i is the third-largest of the channel gains between i and the K receivers. Condition (23) becomes:

$$\sum_{\substack{n=1 \\ n \neq i}}^{N} \frac{h_{n,k}}{\max((h_{n,1}, \cdots, h_{n,K}); d_n)} \alpha_n < 1 \quad \forall i, k \tag{26}$$

Appendix A
Norms and related material

A. Concepts and definitions

Let V denote a vector space (defined at [14, pp. 11-12]).

Definition A.1: A function $f : V \rightarrow \Re$ is called a *semi-norm* on V, if it satisfies:

1) $f(v) \geq 0$ for all $v \in V$ (non-negativity)
2) $f(\lambda v) = |\lambda| \cdot f(v)$ for all $v \in V$ and all $\lambda \in \Re$ (homogeneity)
3) $f(v + w) \leq f(v) + f(w)$ for all $v, w \in V$ (the *triangle inequality*)

Definition A.2: If a semi-norm additionally satisfies $f(v) = 0$ if and only if $v = \theta$ (where θ denotes the zero element of V), then f is called a norm on V and $f(v)$ is usually denoted as $\|v\|$.

Remark A.1: It is a simple matter to show that a function that satisfies properties 2 and 3 of Definition A.1 is convex. Thus, (semi-)norm-minimisation problems are often well-behaved.

Definition A.3: The Hölder norm with parameter $p \geq 1$ ("p-norm") is denoted as $||\cdot||_p$ and defined for $x \in \Re^N$ as

$$\|\mathbf{x}\|_p = (|x_1|^p + \cdots + |x_N|^p)^{\frac{1}{p}} \quad \text{(A.1)}$$

Remark A.2: With $p = 2$, the Hölder norm becomes the familiar Euclidean norm. The $p = 1$ case is also often encountered. Furthermore, it can be shown that $\lim_{p\to\infty} \|\mathbf{x}\|_p = \max(|x_1|, \cdots, |x_N|)$, which leads to the following definition:

Definition A.4: For $x \in \Re^N$, the supremum or infinity norm is denoted as $\|\cdot\|_\infty$ and defined as

$$\|\mathbf{x}\|_\infty := \max(|x_1|, \cdots, |x_N|) \quad \text{(A.2)}$$

Definition A.5: For $\mathbf{v} \in \Re^N$, let $\mathbf{w}$ be such that $w_i := |v_i|$, and denote $\mathbf{w}$ as $|\mathbf{x}|$.

Definition A.6: A norm, $\|\cdot\|$, on $\Re^N$ is called an *absolute vector norm* if it depends only on the absolute values of the components of the vector; that is, for $\mathbf{v} \in \Re^N$, and $\mathbf{w}$ such that $w_i := |v_i|$, $\|\mathbf{v}\| \equiv \|\mathbf{w}\|$.

Definition A.7: For $\mathbf{x}$ and $\mathbf{y} \in \Re^N$, let $\mathbf{x} \leq \mathbf{y}$ mean that $x_i \leq y_i$ for each i. A norm, $\|\cdot\|$, on $\Re^N$ is said to be *monotonic* if, for any $\mathbf{x}$ and $\mathbf{y} \in \Re^N$, $|\mathbf{x}| \leq |\mathbf{y}|$ implies that $\|\mathbf{x}\| \leq \|\mathbf{y}\|$.

B. General technical results

Lemma A.1: (Reverse triangle inequality) If the function $f\colon V \to \Re$ satisfies property 3 of Definition A.1, then $|f(\mathbf{x}) - f(\mathbf{y})| \leq f(\mathbf{x} - \mathbf{y})$.

Proof:

Without loss of generality, suppose that $f(\mathbf{x}) \geq f(\mathbf{y})$ which implies that $f(\mathbf{x}) - f(\mathbf{y}) \equiv |f(\mathbf{x}) - f(\mathbf{y})|$.

Observe that $\mathbf{x} \equiv (\mathbf{x} - \mathbf{y}) + \mathbf{y}$. By hypothesis, $f(\mathbf{x}) \equiv f((\mathbf{x} - \mathbf{y}) + \mathbf{y}) \leq f(\mathbf{x} - \mathbf{y}) + f(\mathbf{y})$ which implies that $f(\mathbf{x}) - f(\mathbf{y}) = |f(\mathbf{x}) - f(\mathbf{y})| \leq f(\mathbf{x} - \mathbf{y})$ ■

Remark A.3: Through Lemma A.1 one can prove that all norms are continuous.

Theorem A.1: A norm on $\Re^N$ is monotonic if and only if it is an absolute vector norm.

Proof: See [15] or [16, p.344]. ■

Theorem A.2: Let $\|\cdot\|$ be a monotonic norm on $\Re^M$ and let T be an $M \times M$ non-singular real matrix. Then, $\|\mathbf{x}\|_T := \|T\mathbf{x}\|$ for $\mathbf{x} \in \Re^M$ defines another monotonic norm on $\Re^M$.

Proof: See [17, Theorem 5.3.2]. ■

Appendix B
Banach fixed-point theory

Definition B.1: A map T from a normed space $(V, \|\cdot\|)$ into itself is a *contraction* if there exists $\lambda \in [0,1)$ such that for all $x, y \in V$, $\|T(x) - T(y)\| \leq \lambda \|x - y\|$.

Definition B.2: (Successive approximation) For expository convenience, let $T^m(x_1)$ for $x_1 \in V$ be defined inductively by $T^0(x_1) = x_1$ and $T^{m+1}(x_1) = T(T^m(x_1))$, with $m \in \{1, 2, \cdots\}$.

Theorem B.1: (Banach's Contraction Mapping Principle) If T is a contraction mapping on a *complete* normed space V, then there is a unique $x^* \in V$ such that $x^* = T(x^*)$. Moreover, x^* can be obtained by successive approximation, starting from an arbitrary initial $x_0 \in V$; i.e., for any $x_0 \in V$, $\lim_{m\to\infty} T^m(x_0) = x^*$

Proof: See [11][12, Theorem 3.1.2, p. 74]. ■

Appendix C
Specific technical results

Lemma C.1: Let $\mathbf{a} = (a_1, \cdots, a_M) \in \Re^M$. Then the function $f(\mathbf{x}) := \sum_{m=1}^M |a_m x_m|$ for $\mathbf{x} \in \Re^M$ satisfies Definition A.1.

Proof:

Only property 3 of Definition A.1 is in doubt. Consider $\mathbf{x}, \mathbf{y} \in \Re^M$.

$f(\mathbf{x} + \mathbf{y}) := \sum_{m=1}^M |a_m(x_m + y_m)| \leq \sum_{m=1}^M (|a_m x_m| + |a_m y_m|) \equiv \sum_{m=1}^M |a_m x_m| + \sum_{m=1}^M |a_m y_m|$ ■

Remark C.1: In Lemma C.1, one can write f as $f(\mathbf{x}) = \|D\mathbf{x}\|_1$ where D is the diagonal matrix $D := \mathrm{diag}(a_1, \cdots, a_M)$, and $\|\cdot\|_1$ denotes the Hölder 1-norm (Definition A.3). If $a_i \neq 0 \quad \forall i$, then D is non-singular, Theorem A.2 applies, and f is a norm.

Lemma C.2: For $\mathbf{x} \in \Re^N$ and $M \leq N$, let $\max(\mathbf{x}; M)$ denote the Mth largest absolute value of the components of x. The function $\max(\mathbf{x}; M)$ satisfies Definition A.1.

Proof:

Only property 3 of Definition A.1 is in doubt. Consider $\mathbf{x}, \mathbf{y} \in \Re^N$.

$$\begin{aligned} \max(\mathbf{x}+\mathbf{y}; M) &\equiv \\ \max\left((|x_1+y_1|, \cdots, |x_N+y_N|); M\right) &\leq \\ \max\left((|x_1|+|y_1|, \cdots, |x_N|+|y_N|); M\right) &\leq \\ \max\left((|x_1|, \cdots, |x_N|); M\right) + \max\left((|y_1|, \cdots, |y_N|); M\right) &\equiv \\ \max(x; M) + \max(y; M) & \end{aligned}$$

■

Lemma C.3: For $\mathbf{x} \in \Re^N$ and $M < n \leq N$, let $\max(\mathbf{x}; M, n)$ denote the Mth largest absolute value of the components of $\mathbf{x}$, while considering *only* the first n components of x. The function $\max(\mathbf{x}; M, n)$ satisfies Definition A.1.

Proof:

Only property 3 of Definition A.1 is in doubt.

Consider $\mathbf{x}, \mathbf{y} \in \Re^N$.

$\max(\mathbf{x}+\mathbf{y}; M, n) \equiv \max\left((|x_1+y_1|, \cdots, |x_n+y_n|); M\right)$.

By Lemma C.2, $\max\left((|x_1+y_1|, \cdots, |x_n+y_n|); M\right) \leq \max\left((|x_1|, \cdots, |x_n|); M\right) + \max\left((|y_1|, \cdots, |y_n|); M\right) \equiv \max(\mathbf{x}; M, n) + \max(\mathbf{y}; M, n)$ ■

Remark C.2: The proof of Lemma C.3 would remain valid if in the definition of $\max(x; M, n)$ the phrase "a given subset of n components" replaces "the first n components".

Remark C.3: Notice that $\max(x; M) \equiv \max(x; M, N)$. $\max(x; M, n)$ (and hence $\max(x; M)$) fails to satisfy Definition A.2, because the Mth largest component of a non-zero vector may be zero (e.g., consider $(1, 0, \ldots, 0)$ and $M = 2$).

Lemma C.4: ("Norm of norms"). Let $\nu_1, \cdots, \nu_K$ be K semi-norms (Definition A.1) on a real vector space V. Let $\mathcal{N}$ be a semi-norm on $\Re^K$ such that $\mathcal{N}(\mathbf{a}) \leq \mathcal{N}(\mathbf{a}+\mathbf{b}) \quad \forall \mathbf{a}, \mathbf{b} \in \Re^K_+$ (i.e., $\mathcal{N}$ is *non-decreasing*). The function $f: V \to \Re$ defined by $f(x) := \mathcal{N}(\nu_1(x), \cdots, \nu_K(x)), \forall x \in V$ is a semi-norm on V.

Proof:

It is evident that f satisfies properties 1 and 2 of Definition A.1. Consider $x, y \in V$ and define $\mathbf{X} := (\nu_1(x), \cdots, \nu_K(x))$ and $\mathbf{Y} := (\nu_1(y), \cdots, \nu_K(y))$. Thus, $f(x) \equiv \mathcal{N}(\mathbf{X})$ and $f(y) \equiv \mathcal{N}(\mathbf{Y})$.
$f(x+y) \equiv \mathcal{N}(\nu_1(x+y), \cdots, \nu_K(x+y))$.
By hypothesis, $\nu_k(x+y) \leq \nu_k(x) + \nu_k(y)\ \forall k \in \{1, \cdots, K\}$.
Since $\mathcal{N}$ is non-decreasing, $\mathcal{N}(\nu_1(x+y), \cdots, \nu_K(x+y)) \leq \mathcal{N}(\nu_1(x)+\nu_1(y), \cdots, \nu_K(x)+\nu_K(y)) \equiv \mathcal{N}(\mathbf{X}+\mathbf{Y})$.
Since $\mathcal{N}$ satisfies the triangle inequality, $\mathcal{N}(\mathbf{X}+\mathbf{Y}) \leq \mathcal{N}(\mathbf{X}) + \mathcal{N}(\mathbf{Y}) \equiv f(x) + f(y) . \therefore \quad f(x+y) \leq f(x) + f(y)$
■

Remark C.4: In Lemma C.4, it is evident that if V is $\Re^N$, and each ν_i is additionally non-decreasing ($\nu_i(\mathbf{x}) \leq \nu_i(\mathbf{x}+\mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \Re^N_+$), then f is additionally non-decreasing on $\Re^N$.

Remark C.5: Theorem B.1 holds for any norm (see Definition A.2). In Lemma C.5 we characterise the conditions under which certain transformation is a contraction under the infinity norm (Definition A.4) (the sub-index of $\|\cdot\|_\infty$ is omitted for notational convenience).

Lemma C.5: Let $\vec{1}_M$ denote the element of $\Re^M$ with each component equal to 1. If for $\mathbf{x} \in \Re^N$ and $i = 1, \ldots, N$, f_i satisfies Definition A.1 as well as (i)monotonicity: $f_i(\mathbf{x}) \leq f_i\left(\|\mathbf{x}\|_\infty \vec{1}_N\right)$ and (ii) $f_i(\vec{1}_N) < 1$, then the transformation $\mathbf{T}$ defined by $T_i(\mathbf{x}) = f_i(\mathbf{x}) + c_i$ for $\mathbf{x} \in \Re^N$, $i = 1 \ldots N$, $c_i \in \Re$ satisfies Definition B.1.

Proof:

Consider $\mathbf{x}, \mathbf{y} \in \Re^N$.

$$\|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})\| = \max \begin{bmatrix} |f_1(\mathbf{x}) - f_1(\mathbf{y})| \\ \vdots \\ |f_N(\mathbf{x}) - f_N(\mathbf{y})| \end{bmatrix} \tag{C.3}$$

By Lemma A.1 (the reverse triangle inequality), $|f_i(\mathbf{x}) - f_i(\mathbf{y})| \leq f_i(\mathbf{x}-\mathbf{y})$. Thus,

$$\max \begin{bmatrix} |f_1(\mathbf{x}) - f_1(\mathbf{y})| \\ \vdots \\ |f_N(\mathbf{x}) - f_N(\mathbf{y})| \end{bmatrix} \leq \max \begin{bmatrix} f_1(\mathbf{x}-\mathbf{y}) \\ \vdots \\ f_N(\mathbf{x}-\mathbf{y}) \end{bmatrix} \tag{C.4}$$

Let $M_{x,y} := \max(|x_1 - y_1|, \cdots, |x_N - y_N|) \equiv \|\mathbf{x}-\mathbf{y}\|$
By monotonicity,

$$f_i(\mathbf{x}-\mathbf{y}) \leq f_i(M_{xy}, \cdots, M_{xy}) \equiv f_i(M_{xy}\vec{1}_N) \tag{C.5}$$

By homogeneity (condition (2) of Definition A.1)

$$f_i(M_{xy}\vec{1}_N) = M_{xy} f_i(\vec{1}_N) \equiv \|\mathbf{x}-\mathbf{y}\| f_i(\vec{1}_N) \tag{C.6}$$

Thus,

$$\|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})\| \leq \lambda \|\mathbf{x}-\mathbf{y}\| \tag{C.7}$$

where $\lambda := \max\{f_1(\vec{1}_N), \cdots, f_N(\vec{1}_N)\} < 1$ ■

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