

How to Position n Transmitter-Receiver Pairs in $n - 1$ Dimensions such that Each Can Use Half of the Channel with Zero Interference from the Others

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Abstract—This work is inspired by the question “Can 100 speakers talk for 30 minutes each in one room within one hour and with zero interference to each other’s audience?” posed by Cadambe and Jafar at the 45th Allerton conference 2007, see [1]. We consider the problem of how many transmitter-receiver pairs can be placed such that each desired link may use half of the channel time free of interference from unintended transmissions. The answer is given in the title: at least n pairs, i.e., $2n$ stations can be positioned in the $(n - 1)$ -dimensional Euclidean space such that complete interference alignment in time is achieved. Regular patterns with equal distances between receivers and transmitters, respectively, are the solution. The basic methodology for achieving this result is borrowed from the field of distance geometry.

I. INTRODUCTION AND MOTIVATION

Interference alignment is a change of paradigm from the common strategy that each transmitter uses power in a way best suited for its intended receiver. With interference alignment power is allocated at each transmitter such that the interference caused to other receivers is minimized in the whole network. In this sense, it is a highly cooperative transmission strategy. The concept of interference alignment is comprehensively introduced in [2], [3].

Conventional power assignment usually forms a stable operating point, namely a Nash equilibrium, however, this point is not optimum from a network perspective, as is shown in [4], [5]. Interference alignment turns out to be globally more efficient although it is not a Nash equilibrium if the objective of each user is to maximize his own rate.

The deterministic channel model yields capacity approximations accurate up to a few bits in certain cases, see [6]. Within this model, under the additional assumption of highly regular channel coefficients, interference alignment that achieves the upper capacity bound can be easily constructed, see [2]. In this work, a fully connected wireless network of K users with symmetric channel coefficients is considered for the AWGN channel. It is shown that for Gaussian noise in the high SNR regime, the capacity per user is approximately one half of the capacity achievable in an interference free network. Basic ingredients for this result are: 1. to consider symbols in a Q -ary representation, 2. to select the even components randomly

from a discrete, w.r.t. addition carry-over free set of reals and let each odd component be zero. 3. The channel coefficients are assumed as $H^{[kj]} = 1$, if $k = j$ and $H^{[kj]} = Q^{-1}$, otherwise. The additive superposition of symbols received at each user k ,

$$Y^{[k]} = \sum_{j=1}^K H^{[kj]} X^{[j]} + Z^{[k]}, \quad (1)$$

then consists of the desired unaltered signal and interfering signals from others shifted by one position to the right, for further details see [2]. Hence, the desired signal can be detected nearly interference free. In the high SNR regime, as power tends to infinity at each transmitter, the quoted capacity result is obtained.

Interference alignment in time can be achieved for n transmitter-receiver pairs whenever the propagation delay of each transmitter to the desired receiver is an odd multiple of the slot length t_0 , and the delay to the unintended receivers is an even multiple of t_0 . All transmitters start transmission simultaneously at time zero and transmit simultaneously over all odd time slots. In this scenario, [1] points out that each transmitter-receiver pair can use half of the channel time with no interference from other transmitters.

Assuming that delay is proportional to Euclidean distance the question is how to locate transmitters and receivers such that the above requirements are fulfilled. In a finite dimensional space this seems to be impossible for large numbers of pairs. In [7], a sophisticated pattern of four transmitter-receiver pairs, eight stations in total, is found in the two-dimensional Euclidean plane such that complete interference alignment is achieved. Besides delay by distance, delaying the beginning of transmission between stations is used as an additional parameter to achieve interference alignment.

In this paper, we show that n transmitter-receiver pairs, i.e., a total number of $2n$ stations, can be placed in an $(n - 1)$ -dimensional Euclidean space such that each unintended link has twice the distance of each desired link, hence allowing for interference alignment by delay. We furthermore determine the coordinates of each station and the distances between

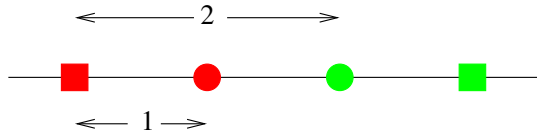


Fig. 1. 2 transmitter-receiver pairs on a line such that perfect interference alignment can be achieved.

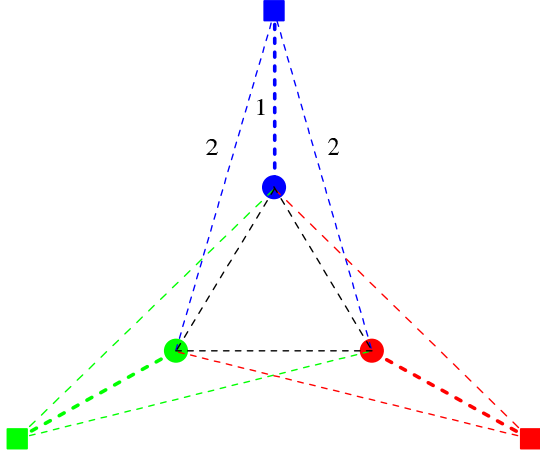


Fig. 2. 3 transmitter-receiver pairs on a plane such that perfect interference alignment can be achieved.

transmitters and receivers, respectively.

In reality, of course only three dimensions are available. Moreover, in real communication scenarios stations are more or less randomly placed, they often move and do not follow regular patterns to allow for interference alignment in time. However, we hope once this simple geometric scenario is understood, further conclusions can be made for interference alignment in space, power or code, where an arbitrary number of dimensions is available.

In the following, the distance between desired links is normalized to 1. Let us start with dimension one. An obvious solution to the above problem is given in Figure 1 where squares indicate transmitters and circles receivers. It is obvious how to extend this solution to two and three dimensions, giving a triangle and a tetrahedron as solutions, respectively, see Figure 2 and 3.

It is an elementary geometric exercise to determine the distance of receivers and transmitters and the coordinates of the corresponding symmetric patterns in two dimensions. It is by no means trivial to extend the result to arbitrary dimensions.

II. NOTATION AND PRELIMINARIES

We first fix the notation throughout this paper. Matrices are denoted by boldface capital letters, vectors by boldface lowercases. \mathbf{I}_n denotes the identity matrix of order n , $\mathbf{1}_n$ the n -vector of ones and $\mathbf{1}_{n \times n}$ the $(n \times n)$ -matrix of ones.

$$\mathbf{E}_n = \mathbf{I}_n - \frac{1}{n} \mathbf{1}_{n \times n}$$

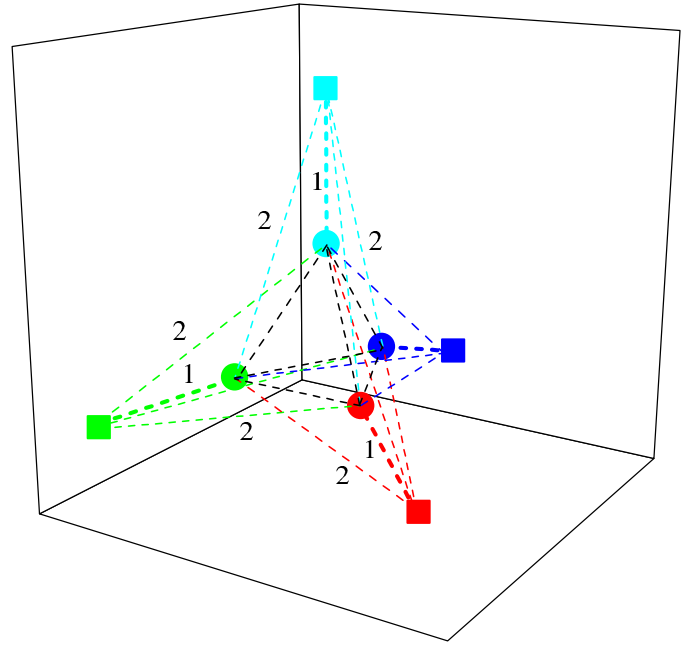


Fig. 3. 4 transmitter-receiver pairs in three-dimensional space such that perfect interference alignment can be achieved.

is the projection onto the orthogonal complement of the diagonal in \mathbb{R}^n , the n -dimensional Euclidean space. $\|\cdot\|$ denotes the Euclidean norm and \mathbf{A}^\top the transpose of some real matrix \mathbf{A} . We often stack vectors and matrices, which is denoted in a straightforward manner by forming block vectors and matrices. It should be mentioned that the rules of matrix multiplication carry over directly to block matrices.

Let $\Delta = (\delta_{ij})_{1 \leq i, j \leq n}$ denote a symmetric matrix of nonnegative entries with zero diagonal, a so called dissimilarity matrix, and $\Delta^{(2)} = (\delta_{ij}^2)_{1 \leq i, j \leq n}$ the matrix of its squared entries. A classical result of distance geometry due to Schoenberg [8] characterizes the set of dissimilarity matrices which allow for a Euclidean embedding.

Theorem 1: Given a dissimilarity matrix $\Delta = (\delta_{ij})_{1 \leq i, j \leq n}$. There are n points $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^k$ such that

$$\delta_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\| \quad (2)$$

if and only if

$$\mathbf{H} = -\frac{1}{2} \mathbf{E}_n \Delta^{(2)} \mathbf{E}_n$$

is nonnegative definite and $\text{rk}(\mathbf{H}) \leq k$. In this case, the rows of any $(n \times k)$ -matrix \mathbf{X} decomposing \mathbf{H} as $\mathbf{H} = \mathbf{X} \mathbf{X}^\top$ may be taken as a configuration satisfying (2).

III. MAIN RESULT

In this section, we construct a regular configuration of n transmitter-receiver pairs in the $(n-1)$ -dimensional Euclidean space such that the distance of desired links is 1 and unintended links have distance 2. We assume that the distance between each transmitter pair is b and between each receiver

pair is a . These assumption lead to the problem of constructing a Euclidean configuration from the block dissimilarity matrix

$$\Delta = \begin{pmatrix} 0 & b & \cdots & b & 1 & 2 & \cdots & 2 \\ b & 0 & \cdots & b & 2 & 1 & \cdots & 2 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ b & b & \cdots & 0 & 2 & 2 & \cdots & 1 \\ 1 & 2 & \cdots & 2 & 0 & a & \cdots & a \\ 2 & 1 & \cdots & 2 & a & 0 & \cdots & a \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 2 & 2 & \cdots & 1 & a & a & \cdots & 0 \end{pmatrix} \quad (3)$$

The upper left block refers to inter-transmitter and the lower right to the pairwise receiver distances.

Block matrix $\Delta^{(2)}$ may now be written as

$$\Delta^{(2)} = \begin{pmatrix} b^2(\mathbf{1}_{n \times n} - \mathbf{I}_n) & -3\mathbf{I}_n + 4\mathbf{1}_{n \times n} \\ -3\mathbf{I}_n + 4\mathbf{1}_{n \times n} & a^2(\mathbf{1}_{n \times n} - \mathbf{I}_n) \end{pmatrix} \quad (4)$$

To be a bit more general we write (4) in its structural form as

$$\Delta^{(2)} = \begin{pmatrix} \alpha_{11}\mathbf{I}_n + \beta_{11}\mathbf{1}_{n \times n} & \alpha_{12}\mathbf{I}_n + \beta_{12}\mathbf{1}_{n \times n} \\ \alpha_{12}\mathbf{I}_n + \beta_{12}\mathbf{1}_{n \times n} & \alpha_{22}\mathbf{I}_n + \beta_{22}\mathbf{1}_{n \times n} \end{pmatrix} \quad (5)$$

with

$$\begin{aligned} \alpha_{11} &= -b^2, & \beta_{11} &= b^2, \\ \alpha_{12} &= -3, & \beta_{12} &= 4, \\ \alpha_{22} &= -a^2, & \beta_{22} &= a^2. \end{aligned} \quad (6)$$

Using the block structure of

$$\mathbf{E}_{2n} = \begin{pmatrix} \mathbf{I}_n - \frac{1}{2n}\mathbf{1}_{n \times n} & -\frac{1}{2n}\mathbf{1}_{n \times n} \\ -\frac{1}{2n}\mathbf{1}_{n \times n} & \mathbf{I}_n - \frac{1}{2n}\mathbf{1}_{n \times n} \end{pmatrix},$$

after tedious but elementary algebra we obtain that

$$\mathbf{E}_{2n}\Delta^{(2)}\mathbf{E}_{2n} = \begin{pmatrix} \alpha_{11}\mathbf{I}_n + \delta_{11}\mathbf{1}_{n \times n} & \alpha_{12}\mathbf{I}_n + \delta_{12}\mathbf{1}_{n \times n} \\ \alpha_{12}\mathbf{I}_n + \delta_{12}\mathbf{1}_{n \times n} & \alpha_{22}\mathbf{I}_n + \delta_{22}\mathbf{1}_{n \times n} \end{pmatrix} \quad (7)$$

with parameters

$$\begin{aligned} \delta_{11} &= -\frac{3\alpha_{11}}{4n} - \frac{\alpha_{12}}{2n} + \frac{\alpha_{22}}{4n} + \frac{\beta_{11}}{4} - \frac{\beta_{12}}{2} + \frac{\beta_{22}}{4}, \\ \delta_{12} &= -\frac{\alpha_{11}}{4n} - \frac{\alpha_{12}}{2n} - \frac{\alpha_{22}}{4n} - \frac{\beta_{11}}{4} + \frac{\beta_{12}}{2} - \frac{\beta_{22}}{4}, \\ \delta_{22} &= -\frac{3\alpha_{22}}{4n} - \frac{\alpha_{12}}{2n} + \frac{\alpha_{11}}{4n} + \frac{\beta_{11}}{4} - \frac{\beta_{12}}{2} + \frac{\beta_{22}}{4}. \end{aligned} \quad (8)$$

In the following a and b are chosen such that $-\frac{1}{2}\mathbf{E}_{2n}\Delta^{(2)}\mathbf{E}_{2n}$ is nonnegative definite and has rank $n-1$, hence allowing for n transmitter-receiver pairs to be embedded in dimension $n-1$.

First,

$$a^2 + b^2 = \frac{8n-6}{n-1} \quad (9)$$

is stipulated. Then by direct computation, using (6) and (8),

$$\alpha_{11} + n\delta_{11} = \alpha_{12} + n\delta_{12} = \alpha_{22} + n\delta_{22} = 0$$

is obtained such that by representation (7)

$$\begin{pmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{0}_n \\ \mathbf{1}_n \end{pmatrix}$$

are eigenvalues of $\mathbf{E}_{2n}\Delta^{(2)}\mathbf{E}_{2n}$ corresponding to eigenvalue 0.

Next, we set

$$a^2b^2 = 9. \quad (10)$$

Let $\mathbf{x} \in \mathbb{R}^n$ be orthogonal to $\mathbf{1}_n$, denoted by $\mathbf{x} \perp \mathbf{1}_n$, and

$$\gamma = \frac{b^2 - 3}{a^2 - 3}. \quad (11)$$

Then

$$\begin{aligned} \begin{pmatrix} \alpha_{11}\mathbf{x} + \alpha_{12}\gamma\mathbf{x} \\ \alpha_{12}\mathbf{x} + \alpha_{22}\gamma\mathbf{x} \end{pmatrix} &= \begin{pmatrix} (-b^2 - 3\frac{b^2-3}{a^2-3})\mathbf{x} \\ (-3 - a^2\frac{b^2-3}{a^2-3})\mathbf{x} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-b^2a^2+9}{a^2-3}\mathbf{x} \\ \frac{-9+a^2b^2}{a^2-3}\mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_n \\ \mathbf{0}_n \end{pmatrix} \end{aligned}$$

such that

$$\begin{pmatrix} \mathbf{x} \\ \gamma\mathbf{x} \end{pmatrix}$$

is an eigenvector of $\mathbf{E}_{2n}\Delta^{(2)}\mathbf{E}_{2n}$ with eigenvalue 0.

From (10) and (11) it follows that

$$b^2 = -3\gamma \text{ and } a^2 = -\frac{3}{\gamma}.$$

Further,

$$\begin{aligned} -\gamma\alpha_{11} + \alpha_{12} &= \gamma b^2 - 3 = -\gamma(a^2 + b^2) \\ -\gamma\alpha_{12} + \alpha_{22} &= 3\gamma - a^2 = -(a^2 + b^2) \end{aligned}$$

such that $\lambda = -(a^2 + b^2)$ is an eigenvalue of $\mathbf{E}_{2n}\Delta^{(2)}\mathbf{E}_{2n}$ corresponding to eigenvector

$$\begin{pmatrix} -\gamma\mathbf{x} \\ \mathbf{x} \end{pmatrix}.$$

In summary we have derived the following result.

Theorem 2: Let a, b satisfy (9) and (10). Then $-\frac{1}{2}\mathbf{E}_{2n}\Delta^{(2)}\mathbf{E}_{2n}$ is nonnegative definite of rank $n-1$. Eigenvalues are $\lambda_1 = 0$ of multiplicity $n+1$ and $\lambda_2 = (a^2 + b^2)/2$ of multiplicity $n-1$.

Let $\mathbf{x}_1, \dots, \mathbf{x}_{n-1} \in \mathbb{R}^n$ be pairwise orthogonal with $\mathbf{x}_i \perp \mathbf{1}_n$, $i = 1, \dots, n-1$, and γ satisfy (11). Then

$$\begin{pmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{pmatrix}, \begin{pmatrix} \mathbf{0}_n \\ \mathbf{1}_n \end{pmatrix}, \begin{pmatrix} \mathbf{x}_1 \\ \gamma\mathbf{x}_1 \end{pmatrix}, \dots, \begin{pmatrix} \mathbf{x}_{n-1} \\ \gamma\mathbf{x}_{n-1} \end{pmatrix}, \begin{pmatrix} -\gamma\mathbf{x}_1 \\ \mathbf{x}_1 \end{pmatrix}, \dots, \begin{pmatrix} -\gamma\mathbf{x}_{n-1} \\ \mathbf{x}_{n-1} \end{pmatrix}$$

is an orthogonal set of eigenvectors, the first $n+1$ corresponding to eigenvalue 0, the latter $n-1$ corresponding to eigenvalue $(a^2 + b^2)/2$.

Hence, n transmitters and n receivers can always be placed in $n-1$ dimensions such that each transmitter has distance 1 to its desired receiver and distance 2 to all unintended receivers.

n	a	b	$(a^2 + b^2)/2$
2	1.0000	3.0000	5.0000
3	1.0705	2.8025	4.5000
4	1.0984	2.7314	4.3333
5	1.1134	2.6945	4.2500
10	1.1403	2.6310	4.1111
100	1.1615	2.5828	4.0101
∞	1.1637	2.5779	4.0000

TABLE I

Conditions (9) and (10) form a set of two equations in two variables which can be easily solved.

For abbreviation purposes let

$$c = \frac{8n - 6}{n - 1}.$$

Then

$$a = \sqrt{\frac{c}{2} - \sqrt{\frac{c^2}{4} - 9}} \quad \text{and} \quad b = \sqrt{\frac{c}{2} + \sqrt{\frac{c^2}{4} - 9}}$$

are a unique solution of (9) and (10).

Obviously, $a = a(n)$ converges to $\sqrt{4 - \sqrt{7}}$ as $n \rightarrow \infty$ and $b = b(n)$ to $\sqrt{4 + \sqrt{7}}$ so that in the limit $n \rightarrow \infty$ the positive eigenvalue λ_2 equals 4.

Table I gives some numerical values of the distance a between receivers, b between transmitters and the positive eigenvalue $(a^2 + b^2)/2$ for different dimensions n and the limit case $n \rightarrow \infty$.

The coordinates of a corresponding transmitter and receiver setup may be computed from the spectral decomposition of $-\frac{1}{2}\mathbf{E}_{2n}\mathbf{\Delta}^{(2)}\mathbf{E}_{2n}$. Let $\frac{1}{\sqrt{n}}\mathbf{1}_n, \mathbf{u}_1, \dots, \mathbf{u}_{n-1} \in \mathbb{R}^n$ be pairwise orthonormal. According to Theorem 1 the rows of the $(2n \times (n-1))$ -matrix

$$\mathbf{X} = \sqrt{\frac{3}{-2\gamma}} \begin{pmatrix} -\gamma\mathbf{u}_1 & \cdots & -\gamma\mathbf{u}_{n-1} \\ \mathbf{u}_1 & \cdots & \mathbf{u}_{n-1} \end{pmatrix}$$

are the coordinates of a configuration of $2n$ points in \mathbb{R}^{n-1} with distance matrix $\mathbf{\Delta}$ from (3).

As an example consider the system of $n-1$ pairwise orthonormal vectors in \mathbb{R}^n , each orthogonal to $\mathbf{1}_n$,

$$\mathbf{u}_k = \frac{1}{\sqrt{k(k+1)}} \left(\underbrace{1, \dots, 1}_k, -k, \underbrace{0, \dots, 0}_{n-k-1} \right)^\top, \quad k = 1, \dots, n-1.$$

It is clear that any other set of $n-1$ pairwise orthonormal vectors, each orthogonal to $\mathbf{1}_n$, provides a valid configuration of receivers and transmitters. This reflects the rotational invariance of the solution in space, and furthermore invariance against orthogonal transformations, e.g., flipping of selected axes.

IV. CONCLUSIONS

We have fully answered the question of how to position $2n$ stations into the $(n-1)$ -dimensional Euclidean space such that complete interference alignment can be achieved. The answer is to extend triangles in two dimensions to tetrahedrons in three, and corresponding structures in higher dimensions. We conjecture that $2n$ is the maximum number embeddable in $(n-1)$ dimensions with all stations starting transmission at the same time. Any change of regularity of the pattern will lead to a smaller number of locations which allow for full interference alignment in one hop. However, if delaying the beginning of transmission between stations is used as an additional degree of freedom, larger numbers of stations may be placed as the example in [7] with four transmitter-receiver pairs in two dimensions shows.

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