On Spatial Patterns of Transmitter-Receiver Pairs that Allow for Interference Alignment by Delay

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***Abstract*— As a theoretical concept, and a means to understand the potential of interference alignment, in this paper we investigate possibilities to place n transmitter-receiver pairs in $n-1$ dimensions such that the interference from unintended transmissions is aligned at each receiving node. By such an arrangement each link has half of the capacity available, at least in theory. Regular patterns of stations are considered. It is shown that placing transmitters "outside" a regular arrangement of receivers provides solutions in any dimension, while placing transmitters "opposite" to intended receivers only yields a solution in dimension one and two. Methodologically we borrow from the field of distance geometry.**

I. Introduction and Motivation

Interference alignment is a highly cooperative strategy to enhance availability of a shared medium for a community of users. It forms a change of paradigm from the common strategy that each receiver uses power in a way best suited for its intended receiver. The key idea is to concentrate interference from unintended links at each receiver such that the desired transmission sees half of the channel free of interference. Hence, everyone gets half the cake. The general concept arose from the study of degrees of freedom of the cross channel and was first introduced in [1]. A survey of interference alignment policies is given in [2], [3].

There are different ways to accomplish interference alignment. If received power at each station is adequate, complete interference alignment can be achieved, as is demonstrated in [2]. A fully connected wireless network of K users with symmetric channel coefficients is considered for the AWGN channel. It is shown that for Gaussian noise in the high SNR regime the capacity per user is approximately one half of the capacity achievable in an interference free network. Symbols are regarded in a Q-ary representation, even components are randomly selected from a discrete, w.r.t. addition carry-over free set of reals, and each odd component is set to zero. The channel coefficients are assumed as $H^{[kj]} = 1$, if $k = j$ and $H^{[kj]} = Q^{-1}$, otherwise. The additive superposition of symbols received at each user k,

$$Y^{[k]} = \sum_{j=1}^{K} H^{[kj]} X^{[j]} + Z^{[k]}, \qquad (1)$$

then consists of the desired unaltered signal and interfering signals from others shifted by one position to the right. Hence, the desired signal can be detected nearly interference free. In the high SNR regime, as power tends to infinity at each transmitter, the quoted capacity result is obtained.

In [4], a MIMO channel of two transmitters and two receivers with mutual interference is investigated. It is shown that the alignment approach is superior to the interference avoidance and iterative water-filling schemes, further highlighting the potential of cooperative interference suppression.

Algorithms for spatial interference in MIMO systems is the theme of the work [3]. Transmit and receive filters are successively updated until alignment is approximately reached. The objective is to minimize the remaining interference power in the dedicated signal at each receiver. Further, the algorithm needs only local channel knowledge and may hence be executed in a decentralized manner. The reciprocity principle of wireless networks constitutes a crucial ingredient.

Ergodic interference alignment is treated in [5]. Complementary channel matrices are used by the transmitters to repeat the same symbol twice. If receivers add the outputs from both channel states all interference is eliminated at unintended receivers while the symbol is doubled at the designated receiver.

Related work is concerned with capacity approximations for wireless networks by investigating *degrees of freedom* [1]. This approach also emphasizes the fact that interference rather than thermal noise will be the bottleneck to the performance of future wireless networks.

Interference alignment in time can be achieved for n transmitter-receiver pairs whenever the propagation delay of each transmitter to the desired receiver is an odd multiple of the slot length t_0, and the delay to the unintended receivers is an even multiple of t_0. All transmitters start transmission at time zero and transmit simultaneously over all odd time slots. In this scenario, the authors [6] point out that each transmitter-receiver pair can use half of the channel time with no interference from other transmitters. The impact of propagation delay on the degrees of freedom in wireless networks is investigated in [7]. The authors give an example of a node placement of four links in the two-dimensional plane that allows perfect interference alignment.

Assuming that delay is proportional to Euclidean distance the question is how to locate transmitters and receivers such that the above requirements are fulfilled. In a finite dimensional space this seems to be impossible for large numbers

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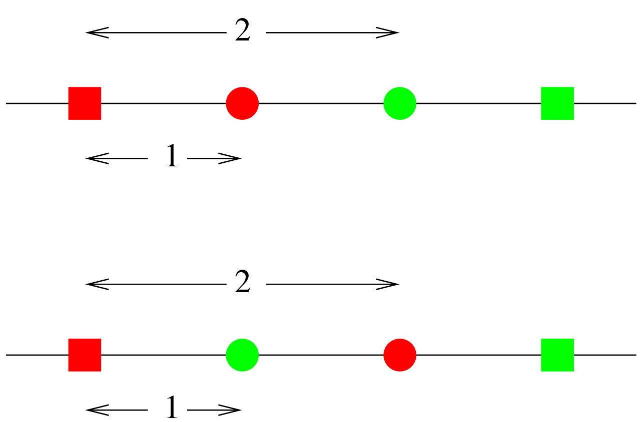

Fig. 1. 2 transmitter-receiver pairs on a line such that perfect interference alignment is achieved. Equal colors indicate designated links.

of pairs. In this paper, we investigate regular patterns of transmitter-receiver pairs in the $(n-1)$-dimensional Euclidean space such that complete interference alignment is accomplished by the above principle. In [8] it is shown that always n transmitter-receiver pairs, i.e., a total number of $2n$ stations, can be placed in an $(n-1)$-dimensional Euclidean space such that each unintended link has twice the distance of each desired link, hence realizing the above described scenario. Other configurations are investigated in the present paper. Interestingly, a type of configuration which exists in one and two dimensions does not have an extension to higher dimensions.

In reality, of course only three dimensions are available. Moreover, in real communication scenarios stations are more or less randomly placed, they often move and do not follow regular patterns to allow for interference alignment in time. However, we hope that once the simple geometric scenario is understood further conclusions can be made for interference alignment by beamforming, power control, coding or by utilizing channel properties, where in each of these cases an arbitrary number of dimensions is available.

In the following, the distance between transmitters and receivers is assumed to be a multiple of the packet duration, which is normalized to be 1.

Let us start with dimension one, i.e., $n = 2$. Obvious solutions to achieve interference alignment are depicted in Figure 1, where squares illustrate transmitters and circles receivers. Nodes of equal color indicate designated links. If transmitters start transmission simultaneously and use only odd slots for transmission, then in the first case intended packets arrive in slots where there is no interference from unintended transmission.

Both structures can be easily extended to two dimensions. The corresponding solutions are two nested triangles, one small triangle of receivers aligned with a larger one of transmitters outside. In the second case, both triangles form a regular hexagon of side length one. Both configurations are shown in Figure 2.

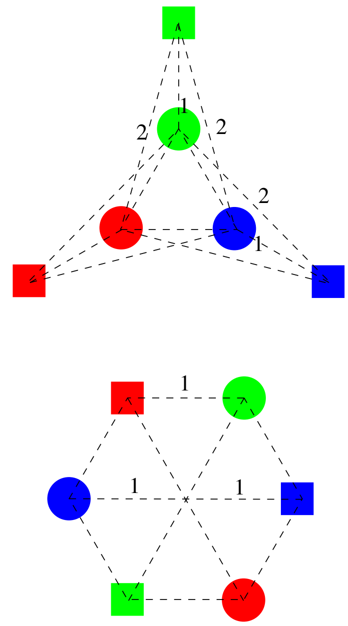

Fig. 2. 3 transmitter-receiver pairs on a plane such that perfect interference alignment is achieved. Equal colors illustrate designated links.

The constellation where unintended links have twice the distance of designated links can be easily extended to three dimensions by aligning two tetrahedrons, the outer representing transmitters, the inner referring to receivers, see Figure 3. The key idea even generalizes to arbitrary many dimensions. Amazingly, there does not exist an analogous extension for the entangled structure of Figure 2, where transmitters are placed "opposite" and not "outside" of designated receivers. It is the key contribution of the present paper to provide the corresponding proof.

II. Notation and Preliminaries

We first fix the notation throughout this paper. Matrices are denoted by boldface capital letters, vectors by boldface lowercases. $\boldsymbol{I}_n$ denotes the identity matrix of order n, $\mathbf{1}_n$ the n-vector of ones and $\mathbf{1}_{n\times n}$ the $(n\times n)$-matrix of ones.

$$\boldsymbol{E}_n = \boldsymbol{I}_n - \frac{1}{n}\mathbf{1}_{n\times n}$$

is the projection onto the orthogonal complement of the diagonal in $\mathbb{R}^n$, the n-dimensional Euclidean space. $\|\cdot\|$ denotes the Euclidean norm and $\boldsymbol{A}^\mathsf{T}$ the transpose of some real matrix $\boldsymbol{A}$. rk$(\boldsymbol{A})$ denotes the rank of matrix $\boldsymbol{A}$. We often stack vectors and matrices, which is denoted in a straightforward manner by forming block vectors and matrices. It should be mentioned that the rules of matrix multiplication carry over directly to block matrices.

Let $\boldsymbol{\Delta} = \big(\delta_{ij}\big)_{1\le i,j\le n}$ denote a symmetric matrix of nonnegative entries with zero diagonal, a so called dissimilarity matrix, and $\boldsymbol{\Delta}^{(2)} = \big(\delta_{ij}^2\big)_{1\le i,j\le n}$ the matrix of its squared entries. A classical result of distance geometry due to Schoenberg [9] characterizes the set of dissimilarity matrices which allow for a Euclidean embedding.

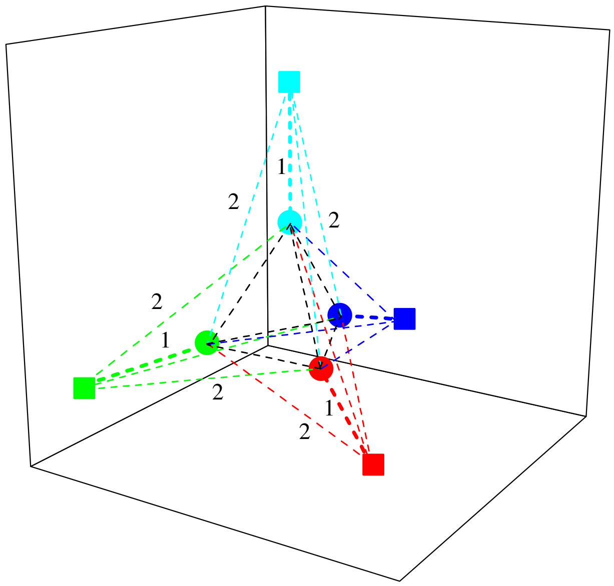

Fig. 3. 4 transmitter-receiver pairs in three-dimensional space such that perfect interference alignment is achieved.

Theorem 1: Given a dissimilarity matrix $\boldsymbol{\Delta} = (\delta_{ij})_{1 \le i,j \le n}$. There are n points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^k$ such that

$$\delta_{ij} = \|\boldsymbol{x}_i - \boldsymbol{x}_j\| \tag{2}$$

if and only if

$$\boldsymbol{H} = -\frac{1}{2}\boldsymbol{E}_n \boldsymbol{\Delta}^{(2)} \boldsymbol{E}_n$$

is nonnegative definite and $\mathrm{rk}(\boldsymbol{H}) \le k$. In this case, the rows of any $(n \times k)$-matrix $\boldsymbol{X}$ decomposing $\boldsymbol{H}$ as $\boldsymbol{H} = \boldsymbol{X}\boldsymbol{X}^\top$ may be taken as a configuration satifying (2).

The results of [8] are now briefly summarized. In general, the distance matrix of the first configurations in Figure 1, 2 and in Figure 3 is of the following type

$$\boldsymbol{\Delta} = \begin{pmatrix} 0 & b & \cdots & b & 1 & 2 & \cdots & 2 \\ b & 0 & \cdots & b & 2 & 1 & \cdots & 2 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ b & b & \cdots & 0 & 2 & 2 & \cdots & 1 \\ 1 & 2 & \cdots & 2 & 0 & a & \cdots & a \\ 2 & 1 & \cdots & 2 & a & 0 & \cdots & a \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 2 & 2 & \cdots & 1 & a & a & \cdots & 0 \end{pmatrix} \tag{3}$$

where the upper left block refers to the distance b between transmitters and the lower right to the distance a between receivers. The off-diagonal blocks comprise the pairwise distances between transmitters and receivers which for the purpose of full interference alignment have to be 1 or 2, respectively. Writing

$$\boldsymbol{\Delta}^{(2)} = \begin{pmatrix} \alpha_{11}\boldsymbol{I}_n + \beta_{11}\mathbf{1}_{n\times n} & \alpha_{12}\boldsymbol{I}_n + \beta_{12}\mathbf{1}_{n\times n} \\ \alpha_{12}\boldsymbol{I}_n + \beta_{12}\mathbf{1}_{n\times n} & \alpha_{22}\boldsymbol{I}_n + \beta_{22}\mathbf{1}_{n\times n} \end{pmatrix} \tag{4}$$

with

$$\begin{aligned} \alpha_{11} &= -b^2, \quad \beta_{11} = b^2, \\ \alpha_{12} &= -3, \quad \beta_{12} = 4, \\ \alpha_{22} &= -a^2, \quad \beta_{22} = a^2. \end{aligned}$$

yields

$$\begin{aligned} &-\frac{1}{2}\boldsymbol{E}_{2n}\boldsymbol{\Delta}^{(2)}\boldsymbol{E}_{2n} \\ &\quad = -\frac{1}{2}\begin{pmatrix} \alpha_{11}\boldsymbol{I}_n + \delta_{11}\mathbf{1}_{n\times n} & \alpha_{12}\boldsymbol{I}_n + \delta_{12}\mathbf{1}_{n\times n} \\ \alpha_{12}\boldsymbol{I}_n + \delta_{12}\mathbf{1}_{n\times n} & \alpha_{22}\boldsymbol{I}_n + \delta_{22}\mathbf{1}_{n\times n} \end{pmatrix} \end{aligned} \tag{5}$$

with parameters

$$\begin{aligned} \delta_{11} &= -\frac{3\alpha_{11}}{4n} - \frac{\alpha_{12}}{2n} + \frac{\alpha_{22}}{4n} + \frac{\beta_{11}}{4} - \frac{\beta_{12}}{2} + \frac{\beta_{22}}{4}, \\ \delta_{12} &= -\frac{\alpha_{11}}{4n} - \frac{\alpha_{12}}{2n} - \frac{\alpha_{22}}{4n} - \frac{\beta_{11}}{4} + \frac{\beta_{12}}{2} - \frac{\beta_{22}}{4}, \\ \delta_{22} &= -\frac{3\alpha_{22}}{4n} - \frac{\alpha_{12}}{2n} + \frac{\alpha_{11}}{4n} + \frac{\beta_{11}}{4} - \frac{\beta_{12}}{2} + \frac{\beta_{22}}{4}. \end{aligned}$$

Now, let

$$c = \frac{8n-6}{n-1},$$

and

$$a = \sqrt{\frac{c}{2} - \sqrt{\frac{c^2}{4} - 9}}, \qquad b = \sqrt{\frac{c}{2} + \sqrt{\frac{c^2}{4} - 9}}.$$

Then, as shown in [8], matrix (5) is nonnegative definite and has rank $n-1$, such that according to Theorem 1 a Euclidean embedding in $n-1$ dimensions exists. Any decomposition

$$-\frac{1}{2}\boldsymbol{E}_{2n}\boldsymbol{\Delta}^{(2)}\boldsymbol{E}_{2n} = \boldsymbol{X}\boldsymbol{X}^\top$$

for some $2n \times (n-1)$-matrix $\boldsymbol{X}$ yields a configuration of transmitters (the first n columns) and receivers (the last n columns) which allows for complete interference alignment.

III. Main Result

In this section, we show that the principle of placing stations as in the second examples of Figure 1 and 2 does not generalize to higher dimensions. This holds true even for dimension three so that there is no complementary construction to the one in Figure 3.

The distance matrix is now assumed to be of type

$$\boldsymbol{\Delta} = \begin{pmatrix} 0 & b & \cdots & b & 2 & 1 & \cdots & 1 \\ b & 0 & \cdots & b & 1 & 2 & \cdots & 1 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ b & b & \cdots & 0 & 1 & 1 & \cdots & 2 \\ 2 & 1 & \cdots & 1 & 0 & a & \cdots & a \\ 1 & 2 & \cdots & 1 & a & 0 & \cdots & a \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & 2 & a & a & \cdots & 0 \end{pmatrix} \tag{6}$$

where a and b have the same interpretation as in (3). Designated packets arrive in even time slots while unintended packets jointly interfere in odd time slots.

$\boldsymbol{\Delta}^{(2)}$ may be written likewise in equation (4), now with parameters

$$\begin{aligned}\alpha_{11} &= -b^2, \quad \beta_{11} = b^2,\\ \alpha_{12} &= 3, \quad \beta_{12} = 1,\\ \alpha_{22} &= -a^2, \quad \beta_{22} = a^2.\end{aligned}$$

$-\frac{1}{2}\boldsymbol{E}_{2n}\boldsymbol{\Delta}^{(2)}\boldsymbol{E}_{2n}$ still has the form (5) with parameters

$$\begin{aligned}\delta_{11} &= -\frac{3\alpha_{11}}{4n} - \frac{\alpha_{12}}{2n} + \frac{\alpha_{22}}{4n} + \frac{\beta_{11}}{4} - \frac{\beta_{12}}{2} + \frac{\beta_{22}}{4},\\ \delta_{12} &= -\frac{\alpha_{11}}{4n} - \frac{\alpha_{12}}{2n} - \frac{\alpha_{22}}{4n} - \frac{\beta_{11}}{4} + \frac{\beta_{12}}{2} - \frac{\beta_{22}}{4},\\ \delta_{22} &= -\frac{3\alpha_{22}}{4n} - \frac{\alpha_{12}}{2n} + \frac{\alpha_{11}}{4n} + \frac{\beta_{11}}{4} - \frac{\beta_{12}}{2} + \frac{\beta_{22}}{4}.\end{aligned}$$

Now, assume that there exists a Euclidean embedding in dimension $n-1$, i.e., there exists some $(2n \times (n-1))$-matrix $\boldsymbol{X}$ such that

$$-\frac{1}{2}\boldsymbol{E}_{2n}\boldsymbol{\Delta}^{(2)}\boldsymbol{E}_{2n} = \boldsymbol{X}\boldsymbol{X}^{\mathsf{T}}. \tag{7}$$

The rows of matrix $\boldsymbol{X}$ are the coordinates of corresponding stations in $\mathbb{R}^{n-1}$. Corresponding to transmitters and receivers two $(n \times (n-1))$-matrices are formed comprising $\boldsymbol{X}$ as

$$\boldsymbol{X} = \begin{pmatrix}\boldsymbol{X}_T\\ \boldsymbol{X}_R\end{pmatrix}.$$

The rows of $\boldsymbol{X}_T$ represent the coordinates of transmitters, and the rows of $\boldsymbol{X}_R$ the coordinates of receivers in $\mathbb{R}^{n-1}$.

Generalizing the principles of the second examples in Figures 1 and 2 to higher dimensions entails the following conditions. Firstly, both transmitters and receivers are centered at the the same point, without loss of generality the origin, hence

$$\mathbf{1}_n^{\mathsf{T}}\boldsymbol{X}_T = \mathbf{0}_{n-1}^{\mathsf{T}} \quad \text{and} \quad \mathbf{1}_n^{\mathsf{T}}\boldsymbol{X}_R = \mathbf{0}_{n-1}^{\mathsf{T}}. \tag{8}$$

Secondly, transmitters are lying in a symmetric fashion opposite of the receivers, formalized by

$$\boldsymbol{X}_T = -\gamma\,\boldsymbol{X}_R \quad \text{for some} \quad \gamma > 0. \tag{9}$$

Recall that

$$\boldsymbol{E}_{2n}\boldsymbol{\Delta}^{(2)}\boldsymbol{E}_{2n} = \begin{pmatrix}\alpha_{11}\boldsymbol{I}_n + \delta_{11}\mathbf{1}_{n\times n} & \alpha_{12}\boldsymbol{I}_n + \delta_{12}\mathbf{1}_{n\times n}\\ \alpha_{12}\boldsymbol{I}_n + \delta_{12}\mathbf{1}_{n\times n} & \alpha_{22}\boldsymbol{I}_n + \delta_{22}\mathbf{1}_{n\times n}\end{pmatrix}.$$

By representation (7) and condition (8) we obtain that

$$\alpha_{11} + n\delta_{11} = \frac{n-1}{4}\left(a^2+b^2\right) - \frac{n+3}{2} = 0$$

Hence, a and b must be chosen to satisfy

$$a^2 + b^2 = \frac{2(n+3)}{n-1}. \tag{10}$$

Furthermore, let the columns of $\boldsymbol{X}_R$ be denoted by

$$\boldsymbol{X}_R = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1})$$

where $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, \mathbf{1}_n$ are pairwise orthogonal. Condition (9) implies that $\begin{pmatrix}-\gamma\boldsymbol{x}_i\\ \boldsymbol{x}_i\end{pmatrix}$ are eigenvectors of $-\frac{1}{2}\boldsymbol{E}_{2n}\boldsymbol{\Delta}^{(2)}\boldsymbol{E}_{2n}$ for some positive eigenvalue α_i, $i = 1, \dots, n-1$.

A complete set of $2n$ orthogonal eigenvectors is hence obtained as

$$\begin{pmatrix}\mathbf{1}_n\\ \mathbf{0}_n\end{pmatrix}, \begin{pmatrix}\mathbf{0}_n\\ \mathbf{1}_n\end{pmatrix}, \begin{pmatrix}\boldsymbol{x}_1\\ \gamma\boldsymbol{x}_1\end{pmatrix}, \dots, \begin{pmatrix}\boldsymbol{x}_{n-1}\\ \gamma\boldsymbol{x}_{n-1}\end{pmatrix},$$
$$\begin{pmatrix}-\gamma\boldsymbol{x}_1\\ \boldsymbol{x}_1\end{pmatrix}, \dots \begin{pmatrix}-\gamma\boldsymbol{x}_{n-1}\\ \boldsymbol{x}_{n-1}\end{pmatrix},$$

where, by the dimensionality constraint the first $(n+1)$ eigenvectors

$$\begin{pmatrix}\mathbf{1}_n\\ \mathbf{0}_n\end{pmatrix}, \begin{pmatrix}\mathbf{0}_n\\ \mathbf{1}_n\end{pmatrix}, \begin{pmatrix}\boldsymbol{x}_1\\ \gamma\boldsymbol{x}_1\end{pmatrix}, \dots, \begin{pmatrix}\boldsymbol{x}_{n-1}\\ \gamma\boldsymbol{x}_{n-1}\end{pmatrix},$$

correspond to eigenvalue 0.

Hence,

$$\begin{aligned}-b^2 + 3\gamma &= 0\\ 3 - \gamma a^2 &= 0\end{aligned}$$

which in turn gives

$$\gamma = \frac{b^2}{3} \quad \text{and} \quad 3 - \frac{a^2b^2}{3} = 0$$

and finally

$$a^2b^2 = 9. \tag{11}$$

The unique solution of (10) and (11) is

$$a_n = \sqrt{\frac{c_n}{2} - \sqrt{\frac{c_n^2}{4} - 9}}$$

and

$$b_n = \sqrt{\frac{c_n}{2} + \sqrt{\frac{c_n^2}{4} - 9}}$$

with

$$c_n = \frac{2(n+3)}{n-1}$$

denoting the right hand side of (10).

Particularly, for $n = 2$ (4 stations on a line) the solution is

$$c_2 = 10 \text{ and } a_2 = 1,\ b_2 = 3,$$

in concert with the second configuration of Figure 1. For $n = 3$ (6 stations on a plane) the values are

$$c_3 = 6 \text{ and } a_3 = b_3 = \sqrt{6},$$

forming a hexagon like in Figure 2.

For $n \geq 4$ it holds that

$$\frac{c_n^2}{4} - 9 < 0$$

such that there is no solution a_n, b_n of (10) and (11) in the real numbers. Summarizing the results so far we have shown the following.

Theorem 2: If $n \geq 4$, n transmitter-receivers pairs cannot be placed in $n-1$ dimensions with distance matrix (6) such that symmetry conditions (8) and (9) hold.

In dimension 1 and 2 however, corresponding configurations are determined above and shown in Figures 1 and 2.

However, as shown in [8], if the distance matrix is given by (3) then an $n-1$ dimensional Euclidean embedding of n transmitter-receiver pairs exists for any $n \geq 2$.

IV. Conclusions

In this paper, we have addressed the question of embedding $2n$ stations, i.e., n designated links of a transmitter-receiver pair each, in the $(n-1)$-dimensional Euclidean space such that complete interference alignment can be achieved by delay. In a previous paper it has been shown that this is always possible by locating stations onto the vertices of triangles, tetrahedrons and higher-dimensional analogues. In dimension one and two there is a complementary construction, yielding a regular hexagon as locations on a plane. Interestingly, this cannot be generalized to dimensions greater than 2, as is shown in the present paper.

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