This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE Globecom 2010 proceedings.

Analytic Framework for the Mutual Information Cumulants of Different MIMO Fading Channels

Paul de Kerret*, Michail Matthaiou†, Rudolf Mathar*, and Josef A. Nossek‡
* Institute for Theoretical Information Technology, RWTH Aachen University,
D-52056 Aachen, Germany, email: {dekerret,mathar}@ti.rwth-aachen.de
† Department of Signals and Systems, Chalmers University of Technology,
Gothenburg, SE-412 96, Sweden, email: michail.matthaiou@chalmers.se
‡ Institute for Circuit Theory and Signal Processing, Technische Universität München (TUM),
Arcistrasse 21, 80333, Munich, Germany, email: nossek@nws.ei.tum.de

***Abstract*—In this paper, we present a general analytical framework for the exact mutual information (MI) cumulants of multiple-input multiple-output (MIMO) systems with perfect receiver channel state information (CSI) and no transmitter CSI. Our derivation is based on a recent parameterization of the joint ordered eigenvalue probability density function (PDF) [1], that encompasses both uncorrelated/semi-correlated Rayleigh channels as well as uncorrelated Rician channels. In addition, we extend our framework to account for the cumulants of doubly-correlated Rayleigh channels and also to deduce tractable expressions in the high Signal-to-Noise ratio (SNR) regime. The cumulants are particularly useful to study all high-order statistics (HOS) of the MI; in fact, they can be used to express the MI mean and variance as a finite sum of determinants. Our analytical expressions are then validated via Monte-Carlo simulations with the attained accuracy being excellent in all cases.**

I. Introduction

In the first contributions on MIMO systems, the prevalent model was the independent and identically distributed (i.i.d.) Rayleigh fading model [2]. However, this model is often acknowledged to be invalid due to either the insufficient spacing between the antenna elements or the limited amount of angular spread, with both phenomena leading to correlated fading. Moreover, a deterministic Line-of-Sight (LoS) component may also be present, in which case the channel matrix entries are more effectively modeled via the Rician distribution. In the following, we will separately consider the effects of correlation and a LoS component. Henceforth, we assume that the receiver has perfect CSI while the transmitter has no CSI. It is then meaningful for the transmitter to isotropically allocate the available power among all the transmit antennas. In this case, exact expressions and tight bounds have first been derived for the i.i.d. Rayleigh channel [2], followed by results for the semi-correlated [3]–[8] and doubly-correlated Rayleigh channels [9]–[11]. The mean MI has also been studied in limiting cases, either in terms of antennas or SNRs [12]–[17].

The study of MIMO Rician channels is more recent and the exact mean MI was derived for uncorrelated channels in [18], [19], but only for the case when the LoS component is either rank-1 or full-rank with distinct LoS eigenvalues. For correlated Rician channels, only bounds are available for arbitrary number of antennas [20]–[24], while some asymptotic results have been provided for infinitely large number of antennas in [15], [25].

The MI moment generating function (MGF) for doubly-correlated Rayleigh channels was derived in [9]–[11], while the moments were obtained, via the cumulants, by means of the trace of polymatrices in [10]. For the uncorrelated Rician case, the MI MGF was given in [11], [19], [26], while the MI non-centered moments of the uncorrelated Rician case were derived in [27] via the unordered eigenvalue PDF.

The main contribution of this paper is to capitalize on a recent framework in [1], in which the joint ordered eigenvalue PDF was derived for uncorrelated/semi-correlated Rayleigh and uncorrelated Rician MIMO channels, and formulate a novel generic framework for the cumulants, for which no closed-form expressions have been reported. By applying this framework in the high-SNR regime, we also derive new simplified approximating formulas for the MI statistics. Moreover, we extend the proposed framework to the case of doubly-correlated Rayleigh channels.

In general, the MI cumulants, κ_n, are particularly insightful for the MI statistics: the first cumulant is equal to the mean, the second to the variance, while the third and fourth cumulants give the skewness γ_1 and the kurtosis γ_2, respectively, via

$$\gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}, \quad \gamma_2 = \frac{\kappa_4}{\kappa_2^2}. \tag{1}$$

The distribution of the MI has been shown to converge to the Gaussian distribution when the number of antennas at both sides grows large under Rayleigh/Rician fading and, more importantly, the Gaussian approximation has been shown to hold even for small number of antennas [6], [19], [20], [25], [26], [28], [29]. The skewness and the kurtosis are inherently interesting because they can be used to test the Gaussianity of a repartition. Indeed, a zero skewness means a symmetric repartition (a positive skewness means a larger right tail), while a zero kurtosis means that the peakness and the tails are identical to those of a Gaussian distribution. Moreover, the cumulants of the Gaussian distribution κ_n are zero for $n > 2$, thus offering another test for the Gaussian approximation.

The remainder of the paper is organized as follows: In Section II, we introduce the expression for the instantaneous MI and the considered channel models. In Section III, we derive the generic analytical framework for the MI cumulants and also present the extension to the doubly-correlated Rayleigh case and the high-SNR regime. A set of numerical results is given in Section IV, while Section V concludes the paper.

Notations : The symbol $(\cdot)^\dagger$ represents the Hermitian transpose operation, $\mathrm{tr}(\cdot)$ yields the matrix trace, and $|\cdot|$ represents the determinant. In some cases, we will write the determinant of a matrix $\mathbf{A}$ in terms of its (i,j)-th elements (e.g. as $|a_{i,j}|$). The symbol $\overset{\mathrm{d}}{\sim}$ denotes "distributed as", while $\sim$ will be used for asymptotic notation. A Vandermonde determinant is defined as $\Delta(\mathbf{x}) = |x_j^{i-1}|$, $\mathrm{E}[\cdot]$ denotes the expectation, and $\delta_{i,j}$ is the Kronecker symbol, equal to 1 if $i = j$ and 0 if $i \neq j$. $\log$ denotes the logarithm with base 2.

II. System Model

A. Mutual Information Expression

We consider a MIMO system with n_R receive antennas and n_T transmit antennas which can be effectively characterized by the MIMO channel matrix $\mathbf{H} \in \mathbb{C}^{n_\mathrm{R}\times n_\mathrm{T}}$. We also define $s \triangleq \min(n_\mathrm{T}, n_\mathrm{R})$, $t \triangleq \max(n_\mathrm{T}, n_\mathrm{R})$ and the instantaneous MIMO correlation matrix $\mathbf{W} \in \mathbb{C}^{s\times s}$ according to

$$\mathbf{W} \triangleq \begin{cases} \mathbf{H}\mathbf{H}^\dagger, & \text{if } n_\mathrm{R} \leq n_\mathrm{T} \\ \mathbf{H}^\dagger\mathbf{H}, & \text{if } n_\mathrm{R} > n_\mathrm{T}. \end{cases} \tag{2}$$

Clearly, the matrix $\mathbf{W}$ has s real, non-negative eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s > 0$ which are concatenated into the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s)^\dagger$. Then, the instantaneous MI is given (in bits/s/Hz) by [2]

$$\mathcal{I} \triangleq \log\left|\mathbf{I}_s + \frac{\gamma}{n_\mathrm{T}}\mathbf{W}\right| = \log\left(\prod_{\ell=1}^{s}(1+\gamma_c\lambda_\ell)\right) \tag{3}$$

with γ being the physically measured SNR at each receiving antenna while $\gamma_c = \gamma/n_\mathrm{T}$.

B. MIMO Channel Models and Wishart Matrices

In the following, we will characterize the distribution of the matrix $\mathbf{W}$ for different MIMO fading channel models. In all cases, the channel is normalized such that $\mathrm{E}[\mathrm{tr}(\mathbf{W})] = n_\mathrm{T} n_\mathrm{R}$.

1) Rayleigh Fading: In this model, there is no LoS component and the channel is assumed to be *doubly-correlated*. For the sake of simplicity, we assume the correlation at the receiver and the transmitter to be separable (or the Kronecker model [3], [20]), and the channel matrix then reads as

$$\mathbf{H} = \mathbf{\Sigma}_\mathrm{R}^{\frac{1}{2}}\mathbf{H}_w\mathbf{\Sigma}_\mathrm{T}^{\frac{1}{2}} \tag{4}$$

where $\mathbf{H}_w \overset{d}{\sim} \mathcal{CN}(0,1)$ (i.i.d. Rayleigh fading matrix), while $\mathbf{\Sigma}_\mathrm{R}$ and $\mathbf{\Sigma}_\mathrm{T}$ are the correlation matrices at the receiver and the transmitter, respectively. We further introduce

$$\mathbf{\Sigma}_s \triangleq \begin{cases} \mathbf{\Sigma}_\mathrm{R}, & \text{if } n_\mathrm{R} \leq n_\mathrm{T} \\ \mathbf{\Sigma}_\mathrm{T}, & \text{if } n_\mathrm{R} > n_\mathrm{T} \end{cases} \quad \mathbf{\Sigma}_t \triangleq \begin{cases} \mathbf{\Sigma}_\mathrm{T}, & \text{if } n_\mathrm{R} \leq n_\mathrm{T} \\ \mathbf{\Sigma}_\mathrm{R}, & \text{if } n_\mathrm{R} > n_\mathrm{T}. \end{cases} \tag{5}$$

The ordered eigenvalues of $\mathbf{\Sigma}_s$ and $\mathbf{\Sigma}_t$ are respectively defined as $\boldsymbol{\sigma}_s = (\sigma_{s,1}, \ldots, \sigma_{s,s})^\dagger$ and $\boldsymbol{\sigma}_t = (\sigma_{t,1}, \ldots, \sigma_{t,t})^\dagger$, and are taken all equal to one, when the fading at the associated end of the link is uncorrelated, or all distinct. The case of some identical eigenvalues can be tackled by taking the limit in the involved expressions but its study is beyond the paper's scope.

If both $\mathbf{\Sigma_s} = \mathbf{I}_s$ and $\mathbf{\Sigma_t} = \mathbf{I}_t$, the fading is said to be *uncorrelated Rayleigh*. Then, $\mathbf{W}$ follows an uncorrelated central Wishart distribution with t degrees of freedom, commonly denoted as $\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{CW}_s(t, \mathbf{I}_s)$.

If $\mathbf{\Sigma_t} = \mathbf{I}_t$, i.e., correlation occurs only on the side with the minimum number of antennas, the channel is said to be *min semi-correlated Rayleigh*. As such, $\mathbf{W}$ is semi-correlated central Wishart denoted as $\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{CW}_s(t, \mathbf{\Sigma}_s)$.

If $\mathbf{\Sigma_s} = \mathbf{I}_s$, i.e., correlation occurs only on the side with the maximum number of antennas, the channel is said to be *max semi-correlated Rayleigh*. We then need to redefine $\mathbf{W}$ by switching the two cases in (2) such that the matrix $\mathbf{W}$ becomes of size $t \times t$ and semi-correlated central pseudo-Wishart (rank-deficient), denoted as $\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{PCW}_t(s, \mathbf{\Sigma}_t)$ [30, Def. 2.3].

2) Uncorrelated Rician Fading: In this model, a deterministic LoS component, $\mathbf{H}_\mathrm{L}$, is present while the random component is uncorrelated. The channel model then reads as

$$\mathbf{H} = \sqrt{\frac{K_r}{K_r+1}}\mathbf{H}_\mathrm{L} + \sqrt{\frac{1}{K_r+1}}\mathbf{H}_w \tag{6}$$

where K_r is the Rician K-factor which represents the ratio between the power of the deterministic component and the scattered waves, and $\mathbf{H}_\mathrm{L}$ satisfies the power constraint $\mathrm{tr}\left(\mathbf{H}_\mathrm{L}^\dagger\mathbf{H}_\mathrm{L}\right) = n_\mathrm{T}n_\mathrm{R}$. We can now define the scaling factor $\varepsilon = 1/\sqrt{K_r+1}$ and the non-centrality matrix $\mathbf{\Omega} \in \mathbb{C}^{s\times s}$ as

$$\mathbf{\Omega} = \begin{cases} K_r\mathbf{H}_\mathrm{L}\mathbf{H}_\mathrm{L}^\dagger, & \text{if } n_\mathrm{R} \leq n_\mathrm{T} \\ K_r\mathbf{H}_\mathrm{L}^\dagger\mathbf{H}_\mathrm{L}, & \text{if } n_\mathrm{R} > n_\mathrm{T} \end{cases} \tag{7}$$

with real eigenvalues $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_s)^\dagger$, and $\omega_1 \geq \ldots \geq \omega_s \geq 0$. Then, $\mathbf{W}$ is uncorrelated noncentral Wishart or $\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{CW}_s(t, \varepsilon\mathbf{I}_s, \mathbf{\Omega})$. To stay consistent with previous studies, we also define $\mathbf{S} = \varepsilon^{-2}\mathbf{W} \overset{\mathrm{d}}{\sim} \mathcal{CW}_s(t, \mathbf{I}_s, \mathbf{\Omega})$ and consider the distribution of the eigenvalues of $\mathbf{S}$ hereafter. Henceforth, for the Rician case we will use $\gamma_c\varepsilon^2$ instead of γ_c and the distribution of $\mathbf{S}$ instead of $\mathbf{W}$ in the MI expression (3).

If the matrix $\mathbf{\Omega}$ is of arbitrary rank L and all its positive eigenvalues are distinct, we denote it as the *arbitrary rank-L* case. This case clearly particularizes to the most important cases of Rician fading, i.e., $L = 1$ for the *rank*-1 case and $L = s$ for the *full-rank* case. If the LoS eigenvalues are all identical and equal to ω, then $\mathbf{\Omega}$ is a scaled identity. The latter case, which will be referred to as the *optimized* LoS MIMO case, occurs when the antenna arrays at both ends are designed to satisfy the design criterion of [31, Eq. (28)].

III. Generic Framework for the MI Cumulants

In this section, we derive analytical expressions for the MI MGF and cumulants for different fading scenarios whose joint ordered eigenvalue PDF can be expressed via the useful

Table I
JOINT EIGENVALUE PDF OF COMPLEX WISHART MATRICES: PARAMETERS AND NORMALIZATION CONSTANTS

Fading scenario	K	$\xi(x)$	$\{\mathbf{\Phi}(\mathbf{x})\}_{i,j}$	$\{\mathbf{\Psi}(\mathbf{x})\}_{i,j}$
Uncorrelated Rayleigh [1]	$\frac{1}{\Gamma_s(s)\Gamma_s(t)}$	$x^{t-s}e^{-x}$	x_j^{i-1}	x_j^{i-1}
Min semi-correlated Rayleigh [1]	$\frac{1}{\Delta(\boldsymbol{\sigma}_s)\Gamma_s(t)\lvert\mathbf{\Sigma}_s\rvert^t}$	x^{t-s}	x_j^{i-1}	$e^{\frac{-x_j}{\sigma_{s,i}}}$
Max semi-correlated Rayleigh [6], [30]	$\frac{1}{\Gamma_s(s)\Delta(\boldsymbol{\sigma}_t)}$	1	x_j^{i-1}	$\begin{cases} \sigma_{t,j}^{t-s-1} e^{\frac{-x_i}{\sigma_{t,j}}}, & \text{if } i \le s \\ \sigma_{t,j}^{i-s-1}, & \text{if } i > s \end{cases}$
Rank-L LoS uncorrelated Rician [33]	$\frac{e^{-\operatorname{tr}(\mathbf{\Omega})}(\Gamma_{s-L}(s-L))^{-1}(-1)_s(-1)_L}{((t-s)!)^L\prod_{\ell=1}^{L}\omega_\ell^{s-L}\prod_{1\le i<j\le L}(\omega_j-\omega_i)}$	$x^{t-s}e^{-x}$	x_j^{i-1}	$\begin{cases} {}_0F_1(t-s+1;\omega_i x_j), & \text{if } i \le L \\ \frac{x_j^{s-i}}{(t-i)!}, & \text{if } i > L \end{cases}$
Optimized LoS uncorrelated Rician [26]	$\frac{(-1)_s e^{-stK_r}}{\Gamma_s(s)}$	$x^{t-s}e^{-x}$	x_j^{i-1}	$\frac{x_j^{s-i}}{(t-i)!}\,{}_0F_1(t-i+1;\omega x_j)$

parameterization of [1]. More specifically, the latter reads as

$$f_{\boldsymbol{\lambda}}(\mathbf{x}) = K|\mathbf{\Phi}(\mathbf{x})||\mathbf{\Psi}(\mathbf{x})|\prod_{\ell=1}^{s}\xi(x_\ell) \tag{8}$$

where the (i,j)-th element of the $s\times s$ matrices $\mathbf{\Phi}(\mathbf{x})$ and $\mathbf{\Psi}(\mathbf{x})$ is defined as $\phi_i(x_j)$ and $\psi_i(x_j)$, respectively. In the case of max semi-correlated Rayleigh fading, $\mathbf{\Psi}(\mathbf{x})$ is of size $t\times t$ and its (i,j)-th element can not be expressed as $\psi_i(x_j)$. The extension of the generic framework to this modified case will be presented in Corollary 1. Please note that this case was briefly discussed in [1], but not explicitly studied.

The parameters K, ξ, $\mathbf{\Phi}(\mathbf{x})$ and $\mathbf{\Psi}(\mathbf{x})$ for the above mentioned fading scenarios are summarized in Table I, where ${}_0F_1(\cdot;\cdot)$ denotes the standard generalized hypergeometric function defined in [32, Eq. (9.14.1)], and $\Gamma_m(n)=\prod_{i=1}^{m}(n-i)!$. For the sake of readability, we have introduced $(-1)_n = (-1)^{n(n-1)/2}$ which represents the signature of the permutation $\sigma(j)=n-j+1$ for $j=1,\ldots,n$. Note that the PDF of the doubly-correlated Rayleigh case can not be written as in (8), but only as an infinite sum of terms written as (8), which fits in the more general framework of [30]. However, this expression is not practical to derive the MI cumulants since it does not lend itself to analytical manipulations. Nevertheless, it will be demonstrated in Theorem 3 that, by using the MGF expression given in [10], it is possible to derive the cumulants in a similar manner as with the general framework.

A. MGF and Cumulants for Arbitrary SNR

Theorem 1: Let the joint ordered PDF of the eigenvalues of $\mathbf{W}$ be given according to (8), then the MI MGF reads

$$\phi_{\text{MGF}}(h) = K|\mathbf{\Lambda}(h)| \tag{9}$$

with $\mathbf{\Lambda}(h)\in\mathbb{R}^{s\times s}$ defined as

$$\{\mathbf{\Lambda}(h)\}_{i,j} = \int_0^\infty \phi_i(u)\psi_j(u)\xi(u)(1+\gamma_c u)^h \mathrm{d}u. \tag{10}$$

Proof: Using (8) and the definition of the MGF yields

$$\phi_{\text{MGF}}(h) = \frac{1}{s!}\int_0^\infty\!\!\cdots\int_0^\infty \prod_{\ell=1}^{s}(1+\gamma_c x_\ell)^h f_{\boldsymbol{\lambda}}(\mathbf{x})\mathrm{d}\mathbf{x} \tag{11}$$

$$= \frac{K}{s!}\int_0^\infty\!\!\cdots\int_0^\infty |\mathbf{\Phi}(\mathbf{x})||\mathbf{\Psi}(\mathbf{x})|\prod_{\ell=1}^{s}\xi(x_\ell)(1+\gamma_c x_\ell)^h \mathrm{d}x_\ell. \tag{12}$$

For (11), we have used the fact that the joint ordered eigenvalue PDF is $s!$ times the unordered one. Substituting (8) into (11) and thereafter using the integral identity [5, Eq. (51)], we can obtain the desired result. ■

Theorem 2: Let the joint ordered PDF of the eigenvalues of $\mathbf{W}$ be given according to (8), the n-th cumulant κ_n of the MI is given by

$$\kappa_n = \sum_{\mathbf{m}} \frac{n!}{m_1!\ldots m_n!}(-1)^{\zeta_n-1}\frac{(\zeta_n-1)!}{|\mathbf{\Lambda}|^{\zeta_n}}\prod_{\ell=1}^{n}\left(\frac{|\mathbf{\Lambda}|^{(\ell)}}{\ell!}\right)^{m_\ell} \tag{13}$$

with $\zeta_n=\sum_{\ell=1}^{n} m_\ell$. We have defined $\sum_{\mathbf{m}}$ as the sum over all the n-uples $(m_1,\ldots,m_n)\in\mathbb{N}^n$ which satisfy $\sum_{\ell=1}^{n}\ell m_\ell = n$, and $|\mathbf{\Lambda}|^{(\ell)}$ as a short-hand notation for $|\mathbf{\Lambda}(h)|^{(\ell)}|_{h=0}$ with $\mathbf{\Lambda}(h)$ defined in (10). $|\mathbf{\Lambda}|^{(\ell)}$ is then given by

$$|\mathbf{\Lambda}|^{(\ell)} = \sum_{\mathbf{k}^\ell}\left|\{\mathbf{\Lambda}\}_{i,j}^{(\alpha_i(\mathbf{k}^\ell))}\right| \tag{14}$$

with $\{\mathbf{\Lambda}\}_{i,j}^{(p)}$ a short-hand notation for $\{\mathbf{\Lambda}(h)\}_{i,j}^{(p)}|_{h=0}$, which can be expressed as

$$\{\mathbf{\Lambda}\}_{i,j}^{(p)} = \int_0^\infty \phi_i(u)\psi_j(u)\xi(u)\left(\log(1+\gamma_c u)\right)^p \mathrm{d}u \tag{15}$$

and $\sum_{\mathbf{k}^\ell}$ the short-hand notation to denote the summation over the multi-index $\mathbf{k}^\ell=(k_1,\ldots,k_\ell)\in\mathbb{N}^\ell, \forall i, k_i\in\{1,\ldots,s\}$. Finally, the function $\alpha_i(\cdot)$ is defined as $\alpha_i(\mathbf{k}^\ell)=\sum_{j=1}^{\ell}\delta_{k_j,i}$.

Proof: Using the Faà-di-Bruno formula for the n-th derivative of a composite function, we obtain

$$\begin{aligned}\kappa_n &= \log(\phi_{\text{MGF}}(h))^{(n)}|_{h=0} \\ &= \sum_{\mathbf{m}} \tfrac{n!}{m_1!\ldots m_n!}\left(\log^{(\zeta_n)}|\mathbf{\Lambda}(0)|\right)\prod_{\ell=1}^{n}\left(\frac{|\mathbf{\Lambda}(h)|_{h=0}^{(\ell)}}{\ell!}\right)^{m_\ell} \\ &= \sum_{\mathbf{m}} \tfrac{n!}{m_1!\ldots m_n!}(-1)^{\zeta_n-1}\frac{(\zeta_n-1)!}{|\mathbf{\Lambda}|^{\zeta_n}}\prod_{\ell=1}^{n}\left(\frac{|\mathbf{\Lambda}|^{(\ell)}}{\ell!}\right)^{m_\ell}\end{aligned} \tag{16}$$

The proof concludes by recalling the well-known result for the ℓ-th derivative derivative of a determinant, that is

$$|\mathbf{\Lambda}(h)|^{(\ell)} = \sum_{\mathbf{k}^\ell}\left|\{\mathbf{\Lambda}(h)\}_{i,j}^{(\alpha_i(\mathbf{k}^\ell))}\right| \tag{17}$$

with $\sum_{\mathbf{k}^\ell}$ defined in Theorem 2. ■

In particular, the mean MI, $\mathrm{E}[\mathcal{I}]$, is obtained by using $n = 1$ in (13) and yields

$$E[\mathcal{I}] = K \sum_{\ell=1}^{s} |\mathbf{\Lambda}_\ell| \tag{18}$$

with $\mathbf{\Lambda}_\ell \in \mathbb{R}^{s\times s}$ defined as

$$\{\mathbf{\Lambda}_\ell\}_{i,j} = \begin{cases} \{\mathbf{\Lambda}\}'_{i,j}, & \text{if } i = \ell \\ \{\mathbf{\Lambda}\}_{i,j}, & \text{if } i \neq \ell. \end{cases} \tag{19}$$

In the following, we consider the case in which the ordered joint PDF is written under the form (8) with $\mathbf{\Phi}(\mathbf{x}) \in \mathbb{R}^{s\times s}$ and $\{\mathbf{\Phi}(\mathbf{x})\}_{i,j} = \phi_i(x_j)$, but the definition of the matrix $\mathbf{\Psi}(\mathbf{x})$ is slightly different. This case can be encountered in a max semi-correlated Rayleigh fading channel.

Corollary 1: Let the joint ordered eigenvalue PDF be given by (8) but $\widetilde{\mathbf{\Psi}}$ be of size $t \times t$ and defined as

$$\{\widetilde{\mathbf{\Psi}}(\mathbf{x})\}_{i,j} = \begin{cases} \psi_j(x_i), & \text{if } i \leq s \\ \psi_{i,j}, & \text{if } i > s. \end{cases} \tag{20}$$

The MGF is then given by

$$\phi_{\text{MGF}}(h) = K|\widetilde{\mathbf{\Lambda}}(h)| \tag{21}$$

with $\widetilde{\mathbf{\Lambda}}(h) \in \mathbb{R}^{t\times t}$ defined as

$$\{\widetilde{\mathbf{\Lambda}}(h)\}_{i,j} = \begin{cases} \int_0^\infty \phi_i(u)\psi_j(u)\xi(u)(1+\gamma_c u)^h \mathrm{d}u, & \text{if } i \leq s \\ \psi_{i,j}, & \text{if } i > s. \end{cases} \tag{22}$$

The cumulants are then given by (13) and (14) with

$$\{\widetilde{\mathbf{\Lambda}}\}^{(n)}_{i,j} = \begin{cases} \int_0^\infty \phi_i(u)\psi_j(u)\xi(u)\left(\log(1+\gamma_c u)\right)^n \mathrm{d}u, & \text{if } i \leq s \\ \psi_{i,j}, & \text{if } i > s. \end{cases} \tag{23}$$

Proof: For the MGF, we can apply [10, Eq. (41)] with $\widetilde{\mathbf{\Lambda}}$ as in the proof of Theorem 1. The derivation of the cumulant is then similar to Theorem 2 because the appended rows at the bottom of the matrix $\widetilde{\mathbf{\Psi}}(\mathbf{x})$ do not depend on h. ■

As was previously mentioned, the case of doubly-correlated Rayleigh MIMO channels admits a similar solution for its MI MGF and cumulants. The corresponding expressions are given by the following theorem:

Theorem 3: The cumulants of the MI in the case of doubly-correlated Rayleigh channels are equal to

$$\kappa_n = c_n + \sum_{\mathbf{m}} \tfrac{n!}{m_1!\ldots m_n!}(-1)^{\zeta_n - 1}\frac{(\zeta_n-1)!}{|\widehat{\mathbf{\Lambda}}|^{\zeta_n}}\prod_{\ell=1}^{n}\left(\frac{|\widehat{\mathbf{\Lambda}}|^{(\ell)}}{\ell!}\right)^{m_\ell} \tag{24}$$

with $c_n = (-1)^n(n-1)!\sum_{\ell=1}^{s-1}\ell^{-(n-1)}$, $\zeta_n = \sum_{\ell=1}^{n} m_\ell$ and $\sum_{\ell=1}^{n}\ell m_\ell = n$. Moreover, the matrix $\widehat{\mathbf{\Lambda}} \in \mathbb{R}^{t\times t}$ is defined as

$$\{\widehat{\mathbf{\Lambda}}\}_{i,j} = \begin{cases} \sigma_{t,j}^{t-s-1}(s-1)!\sum_{k=0}^{s-1}\frac{(\gamma_c\sigma_{s,i}\sigma_{t,j})^k}{(s-k-1)!}, & \text{for } i \leq s \\ \sigma_{t,j}^{i-s-1}, & \text{for } s < i \leq t, \end{cases} \tag{25}$$

while the derivative $|\widehat{\mathbf{\Lambda}}|^{(\ell)}$ is given by

$$|\widehat{\mathbf{\Lambda}}|^{(\ell)} = \sum_{\mathbf{k}^\ell}\left|\{\widehat{\mathbf{\Lambda}}\}^{(\alpha_i(\mathbf{k}^\ell))}_{i,j}\right| \tag{26}$$

where the matrix $\widehat{\mathbf{\Lambda}}^{(p)} \in \mathbb{R}^{t\times t}$ is defined as

$$\{\widehat{\mathbf{\Lambda}}\}^{(p)}_{i,j} = \begin{cases} \frac{\sigma_{t,j}^{t-s-1}}{\sigma_{s,i}}\sum_{k=0}^{s-1}\binom{s-1}{k}\gamma_c^k G^p_{i,j}(k), & \text{for } i \leq s \\ \sigma_{t,j}^{i-s-1}, & \text{for } s < i \leq t. \end{cases}$$

In the above, we have introduced the function

$$G^p_{i,j}(k) \triangleq \int_0^\infty \left(\log(1+\gamma_c u)\right)^p u^k e^{\frac{-u}{\sigma_{s,i}\sigma_{t,j}}}\mathrm{d}u. \tag{27}$$

The summations $\sum_{\mathbf{m}}$ and $\sum_{\mathbf{k}^\ell}$, and the function $\alpha_i(\cdot)$ are defined as in Theorem 2.

Proof: The proof is similar to the proof of Theorem 2 though starting from the MGF expression in [10, Eq. (6)]. ■

B. Generic Framework for the MI Cumulants in the High-SNR Regime

In the high-SNR regime, the MI in (3) is approximated as

$$\mathcal{I} \sim s\log(\gamma_c) + \log|\mathbf{W}|. \tag{28}$$

For the doubly-correlated Rayleigh MIMO channels, the MI is further simplified in the high-SNR regime according to

$$\mathcal{I} \sim s\log(\gamma_c) + \log|\mathbf{\Sigma}_s| + \log\left|\mathbf{H}\mathbf{\Sigma}_t\mathbf{H}^\dagger\right|. \tag{29}$$

Clearly, the centered moments (and the cumulants) are independent of the correlation matrix $\mathbf{\Sigma_s}$ which only creates a shift in the MI. Therefore, we can compute the cumulants for $\mathbf{\Sigma}_s = \mathbf{I}_s$, and extend the results to the case when $\mathbf{\Sigma}_s \neq \mathbf{I}_s$ by adding the shift due to $\mathbf{\Sigma}_s$ in the mean MI according to (29).

Corollary 2: Let the joint ordered PDF of the eigenvalues of $\mathbf{W}$ be given according to (8), the MGF of the MI is approximated in the high-SNR regime by

$$\phi_{\text{MGF}}(h) \sim K\gamma_c^{sh}|\mathbf{\Lambda}_\infty(h)| \tag{30}$$

with $\mathbf{\Lambda}_\infty(h) \in \mathbb{R}^{s\times s}$ defined as

$$\{\mathbf{\Lambda}_\infty(h)\}_{i,j} = \int_0^\infty \phi_i(u)\psi_j(u)\xi(u)u^h\mathrm{d}u. \tag{31}$$

As a consequence, the n-th cumulant κ_n of the MI is approximated in the high-SNR regime by

$$\kappa_n \sim s\log(\gamma_c)\delta_{1,n} + \sum_{\mathbf{m}}\tfrac{n!}{m_1!\ldots m_n!}\tfrac{(-1)^{\zeta_n-1}(\zeta_n-1)!}{|\mathbf{\Lambda}_\infty|^{\zeta_n}}\prod_{\ell=1}^{n}\left(\frac{|\mathbf{\Lambda}_\infty|^{(\ell)}}{\ell!}\right)^{m_\ell} \tag{32}$$

and the ℓ-th derivative of $|\mathbf{\Lambda}_\infty(h)|$ obtained via

$$|\mathbf{\Lambda}_\infty|^{(\ell)} = \sum_{\mathbf{k}^\ell}\left|\int_0^\infty \phi_i(u)\psi_j(u)\xi(u)\left(\log(u)\right)^{\alpha_i(\mathbf{k}^\ell)}\mathrm{d}u\right|. \tag{33}$$

Proof: Proof follows by taking γ_c large in Thms 1-2. ■

C. Analytical Expressions

The doubly-correlated Rayleigh channel has been discussed in Theorem 3 and the results for the other fading scenarios are obtained by substituting the parameters of Table I in Theorems 1 and 2. More importantly, the final expressions can always be written in closed-form by using the integral formulas given in [19, Appendix A], and for the high-SNR

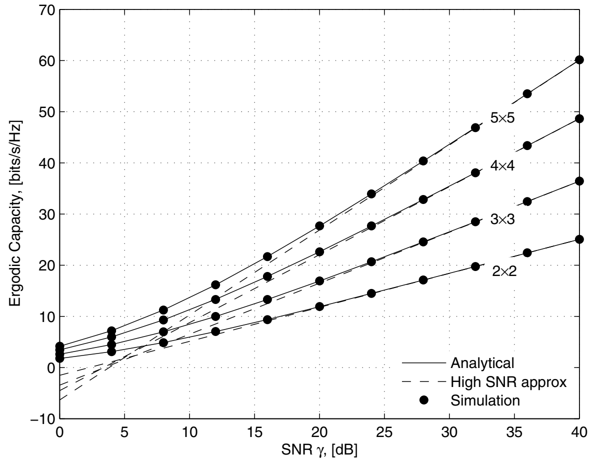

Figure 1. Analytical, high-SNR approximation and simulated MI mean for different Rician MIMO configurations ($K_r = 3$ dB).

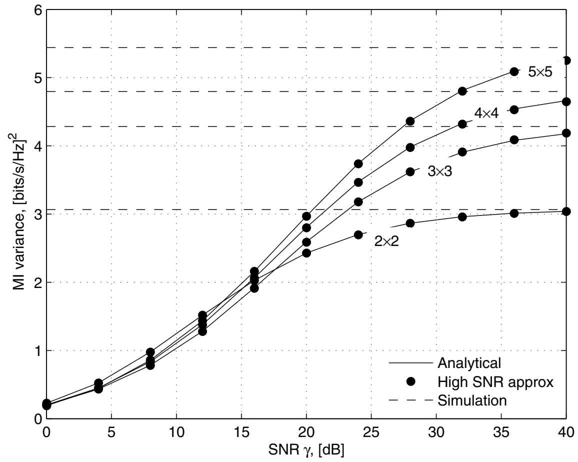

Figure 2. Analytical, high-SNR approximation and simulated MI variance for different Rician MIMO configurations ($K_r = 3$ dB).

approximation, the integral expressions for the Gamma function and its derivatives. Due to space constraints, the resulting closed-form expressions can not be given here.

As an insightful example, we now present the exact mean MI expression for uncorrelated Rayleigh MIMO channels, consistent with the expressions presented in [8], [19], followed by the high-SNR approximation,

$$\begin{aligned} \mathrm{E}[\mathcal{I}] &= \frac{e^{\frac{1}{\gamma_c}} \sum_{\ell=1}^{s} |\mathbf{\Lambda}_\ell|}{\Gamma_s(t)\Gamma_s(s)} \\ \mathrm{E}[\mathcal{I}] &\sim s\log(\gamma_c) + \frac{\sum_{\ell=1}^{s} |\mathbf{\Lambda}_{\infty,\ell}|}{\Gamma_s(t)\Gamma_s(s)}, \quad (\gamma_c \to \infty) \end{aligned} \tag{34}$$

with $\mathbf{\Lambda}_\ell \in \mathbb{R}^{s\times s}$ and $\mathbf{\Lambda}_{\infty,\ell} \in \mathbb{R}^{s\times s}$ respectively defined as

$$\{\mathbf{\Lambda}_\ell\}_{i,j} = \begin{cases} \Gamma(t-s+j+i-1) \sum_{\ell=1}^{t-s+i+j-1} \mathbb{E}_\ell\left(\frac{1}{\gamma_c}\right), & \text{if } i = \ell \\ \Gamma(t-s+j+i-1) & \text{, if } i \neq \ell \end{cases} \tag{35}$$

$$\{\mathbf{\Lambda}_{\infty,\ell}\}_{i,j} = \begin{cases} \Gamma'(t-s+j+i-1), & \text{if } i = \ell \\ \Gamma(t-s+j+i-1), & \text{if } i \neq \ell \end{cases} \tag{36}$$

where $\mathbb{E}_\ell(x) \triangleq \int_1^\infty u^{-\ell} e^{-xu} \mathrm{d}u$ is the exponential integral function.

IV. Simulation Results

In this section, the theoretical analysis is validated via Monte-Carlo simulations. To this end, we have generated $200,000$ realizations of the random matrix $\mathbf{H}$ according to the system model in Section II. A full-rank Rician MIMO system with $K_r = 3$ dB is considered for different number of antennas, $n_\mathrm{R} = n_\mathrm{T} = n$ with $n = 2, 3, 4, 5$.

The MI mean and variance are plotted in Fig. 1 and 2, where we can observe the linear capacity increase with the number of antennas [2], [4] as well as that the high-SNR variance analytical expression is independent of the SNR, which is consistent with the results in [20], [22], [26]. The MI skewness and kurtosis are respectively plotted in Fig. 3 and 4, where it is clearly seen that the skewness and kurtosis tend to zero as the number of antennas n increases, thereby validating the common Gaussian approximation [4], [6], [29]. Moreover, the kurtosis and the skewness are close to zero at low SNR for all cases, while the values at higher SNR depend largely on the antenna configuration, and get closer to zero when the number of antennas increases.

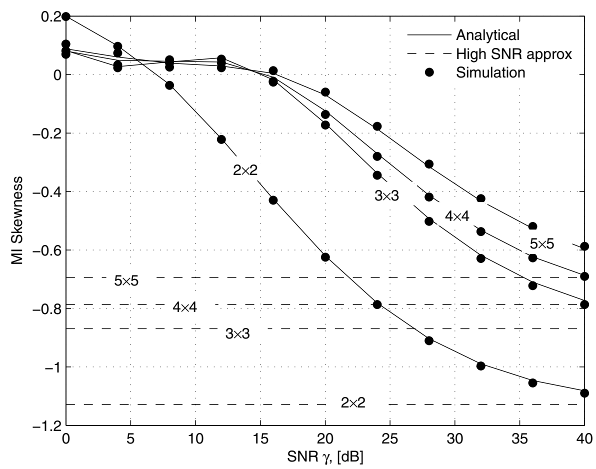

Figure 3. Analytical, high-SNR approximation and simulated MI skewness for different Rician MIMO configurations ($K_r = 3$ dB).

Finally, we can observe an exact agreement between the simulations and the analytical results in all cases considered.

V. Conclusion

In this paper, we have derived a generic analytical framework for the MI cumulants, which enables us to explore not only the MI mean and variance but also its HOS such as the skewness and the kurtosis. The proposed framework is valid for uncorrelated/single-sided correlated Rayleigh channels and

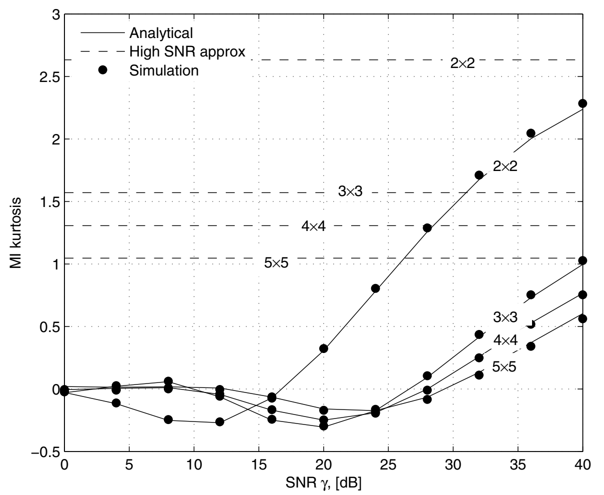

Figure 4. Analytical, high-SNR approximation and simulated MI kurtosis for different Rician MIMO configurations ($K_r = 3$ dB).

arbitrary-rank uncorrelated Rician channels, and has been extended to the case of doubly-correlated Rayleigh channel. Moreover, we have used the framework to get simple high-SNR approximations for the MI cumulants.

Our numerical results demonstrate an excellent match between theory and Monte-Carlo simulations, and give valuable insights into the influence of the model parameters on the HOS, as for example the interesting variation of the skewness and kurtosis in terms of SNR. The proposed framework is further used to test the common Gaussian approximation for the MIMO MI as a function of the number of antennas.

REFERENCES

[1] A. Zanella, M. Chiani, and M. Z. Win, "On the marginal distribution of the eigenvalues of Wishart matrices," *IEEE Trans. Commun.*, vol. 57, no. 4, pp. 1050–1060, Apr. 2009.

[2] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *Europ. Trans. Telecommun.*, vol. 10, no. 6, pp. pp. 585–595, Nov./Dec 1999.

[3] H. Shin and J. H. Lee, "Capacity of multiple-antenna fading channels: Spatial fading correlation, double scattering, and keyhole," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2636–2647, Oct. 2003.

[4] M. Kang and M. -S. Alouini, "Impact of correlation on the capacity of MIMO channels," in *Proc. IEEE Intern. Conf. Commun. (ICC)*, vol. 4, Anchorage, AL, May 2003, pp. 2623–2627.

[5] M. Chiani, M. Z. Win, and A. Zanella, "On the capacity of spatially correlated MIMO Rayleigh-fading channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2363–2371, Oct. 2003.

[6] P. J. Smith, S. Roy, and M. Shafi, "Capacity of MIMO systems with semicorrelated flat fading," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2781–2787, Oct. 2003.

[7] G. Alfano, A. M. Tulino, A. Lozano, and S. Verdú, "Capacity of MIMO channels with one-sided correlation," in *Proc. IEEE Intern. Symp. Spread Spectrum Techn. Appl. (ISSSTA)*, Sydney, Australia, Aug. 2004.

[8] M. R. McKay, I. B. Collings, and A. M. Tulino "Achievable sum rate of MIMO MMSE receivers: A general analytic framework," *IEEE Trans. Inf. Theory*, vol. 56, no. 1, pp. 396–410, Jan. 2010.

[9] M. Kiessling, "Unifying analysis of ergodic MIMO capacity in correlated Rayleigh fading environments," *Europ. Trans. Telecommun.*, vol. 1, no. 1, pp. 17–35, Jan. 2005.

[10] H. Shin, M. Z. Win, J. H. Lee, and M. Chiani, "On the capacity of doubly correlated MIMO channels," *IEEE Trans. Wireless Commun.*, vol. 5, no. 8, pp. 2253–2265, Aug. 2006.

[11] S. H. Simon and A. L. Moustakas, " Capacity and character expansions: Moment-generating function and other exact results for MIMO correlated channels," *IEEE Trans. Inf. Theory*, vol. 52, no. 12, pp. 5336–5351, Dec. 2006.

[12] A. L. Moustakas, S. H. Simon, and A. M. Sengupta, "MIMO capacity through correlated channels in the presence of correlated interferers and noise: A (not so) large N analysis," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2545–2561, Oct. 2003.

[13] C. Martin and B. Ottersten, "Asymptotic eigenvalue distributions and capacity for MIMO channels under correlated fading," *IEEE Trans. Wireless Commun.*, vol. 3, no. 4, pp. 1350–1359, Jul. 2004.

[14] H. Shin, M. Z. Win, and M. Chiani, "Asymptotic statistics of mutual information for doubly correlated MIMO channels," *IEEE Trans. Wireless Commun.*, vol. 7, no. 2, pp. 562–573, Feb. 2008.

[15] A. M. Tulino, A. Lozano, and S. Verdú, "Impact of antenna correlation on the capacity of multiantenna channels," *IEEE Trans. Inf. Theory*, vol. 51, no. 7, pp. 2491–2509, Jul. 2005.

[16] A. Lozano, A. M. Tulino, and S. Verdú, "Multiple-antenna capacity in the low-power regime," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2527–2544, Oct. 2003.

[17] ——, "High-SNR power offset in multiantenna communication," *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4134–4151, Dec. 2005.

[18] S. K. Jayaweera and H. V. Poor, "On the capacity of multiple-antenna systems in Rician fading," *IEEE Trans. Wireless Commun.*, vol. 4, no. 3, pp. 1102–1111, May 2005.

[19] M. Kang and M. -S. Alouini, "Capacity of MIMO Rician channels," *IEEE Trans. Wireless Commun.*, vol. 5, no. 1, pp. 112–122, Jan. 2006.

[20] M. R. McKay and I. B. Collings, "General capacity bounds for spatially correlated Rician MIMO channels," *IEEE Trans. Inf. Theory*, vol. 51, no. 9, pp. 3121–3145, Sept. 2005.

[21] ——, "Improved general lower bound for spatially-correlated Rician MIMO capacity," *IEEE Commun. Lett.*, vol. 10, no. 3, pp. 162–164, Mar. 2006.

[22] M. R. McKay, "Random matrix theory analysis of multiple antenna communication systems," Ph.D. dissertation, School of Electrical and Information Engineering, University of Sydney, 2006.

[23] S. Jin, X. Gao, and X. You, "On the ergodic capacity of rank-1 Rician-fading MIMO channels," *IEEE Trans. Inf. Theory*, vol. 53, no. 2, pp. 502–517, Feb. 2007.

[24] Y. Zhu, P. Y. Kam, and Y. Xin, "On the mutual information distribution of MIMO Rician fading channels," *IEEE Trans. Commun.*, vol. 57, no. 5, pp. 1453–1462, May 2009.

[25] G. Taricco, "Asymptotic mutual information statistics of separately correlated Rician fading MIMO channels," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3490–3504, Aug. 2008.

[26] M. Matthaiou, P. de Kerret, G. K. Karagiannidis, and J. A. Nossek, "Mutual information statistics and beamforming performance analysis of optimized LoS MIMO systems," *in press IEEE Trans. Commun.*

[27] A. Maaref and S. Aïssa, "Joint and marginal eigenvalue distributions of (Non)central complex Wishart matrices and PDF-based approach for characterizing the capacity statistics of MIMO Rician and Rayleigh fading channels," *IEEE Trans. Wireless Commun.*, vol. 6, no. 10, pp. 3607–3619, Oct. 2007.

[28] Z. Wang and G. B. Giannakis, "Outage mutual information of space-time MIMO channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 4, pp. 657–662, Apr. 2004.

[29] B. M. Hochwald, T. L. Marzetta, and V. Tarokh, "Multiple-antenna channel hardening and its implications for rate feedback and scheduling," *IEEE Trans. Inf. Theory*, vol. 50, no. 9, pp. 1893–1909, Sept. 2004.

[30] L. G. Ordóñez, D. P. Palomar, and J. R. Fonollosa, "Ordered eigenvalues of a general class of Hermitian random matrices with application to the performance analysis of MIMO systems," *IEEE Trans. Signal Process.*, vol. 57, no. 2, pp. 672–689, Feb. 2009.

[31] I. Sarris and A. R. Nix, "Design and performance assessment of high-capacity MIMO architectures in the presence of a line-of-sight component," *IEEE Trans. Veh. Techn.*, vol. 56, no. 4, pp. 2194–2202, Jul. 2007.

[32] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Seventh Ed. Academic Press, San Diego, 2007.

[33] S. Jin, M. R. McKay, X. Gao, and I. B. Collings, "MIMO multichannel beamforming: SER and outage using new eigenvalue distributions of complex noncentral Wishart matrices," *IEEE Trans. Commun.*, vol. 56, no. 3, pp. 424–434, Mar. 2008.