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# Max-Min Greedy Interference Alignment on Linear Deterministic $K$-User Interference Channels

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***Abstract*—We explore an algorithmic approximation of achievable lower bounds for the generalized degrees of freedom on finite-field linear deterministic $K$-user interference channels. With a heuristic greedy algorithm, derived from results on the maximum independent set problem in graph theory, max-min fair generalized degrees of freedom for desired links are approached. The idea of interference alignment is implicitly contained in the algorithm. It is applied to linear deterministic interference channels with entirely symmetric cross-channel gains, and compared to the generalized degrees of freedom of a currently known coding scheme. The greedy heuristic also generalizes to deterministic interference channels with asymmetric channel gains.**

## I. Introduction

In wireless communications, the impact of co-channel interference has been an elementary bottleneck in the query for high data rates. The characterization of the *capacity for the $K$-user Gaussian interference channel* is a central problem and has been open over several decades.

In order to approximate this capacity within a limited number of bits, a wireless channel can be characterized in the high-SNR regime by the *finite-field linear deterministic channel model* introduced by Avestimehr, Diggavi and Tse in [1]. The underlying concept is to eclipse the impact of noise and to highlight the influence of wireless *broadcast*, *interference* and *signal scale* in the shared medium. This model is useful to investigate the quantity of the *generalized degrees of freedom* (GDoF) as introduced in [2]. Such an approach provides a first insight into the behaviour of interfering communications in wireless multiple-user channels by allowing a traceable analysis of structured coding strategies [3].

An upcoming concept to deal with co-channel interference has been accomplished by Maddah-Ali, Motahari and Khandani in [4] on Gaussian MIMO $X$-channels. It was further generalized by Cadambe and Jafar in their awarded publication [5] for Gaussian $K$-user MIMO interference channels. Their novel idea of *interference alignment* addresses the possibility to overlap the interfering signals from undesired transmitters within a dedicated subspace at each receiver ensuring a disjoint subspace for the desired signal. An integral aspect of this method is to cooperatively coordinate the available signalling dimensions at both transmitters and receivers. The goal is to resolve the restricting interference-limitation in a multiple-user network so that all undesired transmitters behave like a single interferer. Then, each user can simultaneously achieve about $1/2$ degrees of freedom instead of only $1/K$ degrees of freedom achieved by orthogonal schemes.

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Interference alignment can be applied to the deterministic channel model as well. An initial scheme for the characterization of the achievable rates in deterministic $K$-user interference channels is provided in [6]. The GDoF of the *symmetric* $K$-user interference channel are addressed in [7]. It is closely related to the symmetric two-user interference channel treated in [2]. Linear coding strategies using interference alignment are proposed for the symmetric $X$-channel in [8].

In the present paper, finite-field linear deterministic interference channels for $K$ users serve as a model to analyze the GDoF of coding schemes applying the idea of interference alignment. Coding schemes, that have been provided in the literature [6], [7], can only operate on symmetric channels. However, these schemes do not translate to asymmetric channels. To tackle this problem, an alternative approach for coding is motivated by algorithmic methods of graph theory [9]. Inspired by [10], [11], the communication problem is narrowed down to a *constricted maximum independent set problem*.

The key idea provided here concerns a greedy approach for interference alignment on a contracted graph model deduced from the deterministic model. The corresponding coding scheme yields congruent results to those given in [7] for the symmetric $K$-user interference channel. An analytical expression for the GDoF is derived. The opportunity to apply the greedy algorithm to asymmetric interference channels is straightforward. Hence, a simple tool to perform interference alignment is given for asymmetric interference channels.

The present paper is organized as follows: We commence a review of the linear deterministic model for the $K$-user interference channel, introduce the contracted graph representation and the constricted maximum independent set problem in Section II. The greedy algorithm is proposed in Section III. It is evaluated in Section IV for two examples of deterministic interference channels. In Section V, we give a conclusion.

## II. System Model

### A. The Linear Deterministic $K$-User Interference Channel

**Definition 1.** *The input-output equation of the* ***fully connected finite-field linear deterministic*** $K$***-user interference channel*** *is a linear transformation of signal vectors in* $\mathbb{F}_2^q$ *(cf. [1]):*

$$\mathbf{y}_j \quad = \quad \sum_{m=1}^{K} \mathbf{S}^{q-n_{m,j}} \mathbf{x}_m, \ \forall j \in \mathcal{K}. \qquad (1)$$

There are $K$ pairs of transmitters and receivers sharing a common index in the set $\mathcal{K} = \{1, \ldots, K\}$. The vector $\mathbf{y}_j \in \mathbb{F}_2^q$, with length $q$, is the received output signal at receiver $j \in \mathcal{K}$. The input signal vector of transmitter $m \in \mathcal{K}$ is denoted by $\mathbf{x}_m \in \mathbb{F}_2^q$. In order to model signal scale in a wireless scenario, the channel is represented by a *linear transformation matrix* chosen as a *downshift matrix* $\mathbf{S}$ of size $q \times q$ with elements $s_{m,j}$ denoting the $m$-th row and $j$-th column. The elements are constrained to $s_{m+1,m} = 1$ for $m = 1,\ldots,q-1$ and otherwise to $s_{m,j} = 0$. The non-negative exponent $q - n_{m,j}$ determines the constant *channel gain* $n_{m,j}$ between transmitter $m$ and receiver $j$. The $l$-th *level* of a signal is simply the $l$-th element of the considered signal vector, e.g. $\mathbf{x}_m(l)$, $l \in \mathbb{N}$. The $n_{m,j}$ most significant levels from transmitter $m$ are exactly mapped onto the $n_{m,j}$ least significant levels of receiver $j$. All other levels, received below the least significant level, are considered as noise and hence truncated. The point-to-point channel of a $(m, j)$-pair of transmitters and receivers is now termed as a *link*. Desired links are assumed to have equal channel gains, i.e. $n_{m,m} = q$ for all $m \in \mathcal{K}$.

The term *fully connected* indicates that each transmitter $m$ causes interference to each undesired receiver $j \neq m$. Accordingly, there are $K^2$ links in total for $K$ desired links in the channel. This property is guaranteed by taking the size of signal vectors as $q = \max_{m,j}(n_{m,j})$, i.e. $n_{m,j} \leq q$. Summation and multiplication is performed in $\mathbb{F}_2$. The influence of wireless broadcast at each transmitter and of interference at each receiver is inherent in the model due to summation of all received signals on the same levels. Assuming constant channel gains, a time-index is omitted to keep notation simple.

An initial coding scheme for interference alignment is proposed in [6] for a basic channel example with $n_{m,j} = q - 1$ for all interfering links $m \neq j$. There, a subset of signal levels is selected for transmission (all even numbered levels) and no information is conveyed on the remaining levels (all odd numbered levels). This is possible because of the very simple structure of that considered channel. It is shown for $K$ desired links with $N+1$ levels each, that a rate of $\frac{N+1}{2}$ is achievable.

In the following, the term of an *active* level indicates that this level is used to transmit/decode symbols at the related transmitter/receiver. An *inactive* level corresponds to either conducting no transmission or to discard incoming signals.

As introduced in [1], an equivalent graph representation establishes a clear intuition of the deterministic channel model. An *undirected graph* $G$ is completely defined by an ordered tuple $G = (V, E)$ with the set of *vertices* $V$ and the set of *edges* $E \subseteq V \times V$. Transmitters $T_m$ and receivers $R_j$ are represented as *supernodes*, each denoted as a disjoint set of vertices $T_m, R_j \subset V$, with $T_m \cap R_j = \emptyset$ for all $m, j \in \mathcal{K}$. These vertices represent the discrete signal levels so that each supernode contains $|T_m| = |R_j| = q$ vertices. The cardinality of a set $S$ denoted as $|S|$. Each mapping between these signal levels contained in the linear transformation $\mathbf{S}^{q-n_{m,j}}$ corresponds to edges $e = (v_m, v_j) \in E$. An example of a fully connected linear deterministic interference channel is depicted in Fig. 1 for $K = 3$ direct links with normalized channel gains and some arbitrary gains given for the cross-links.

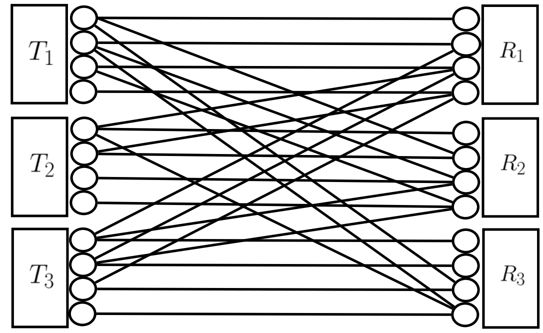


Fig. 1. An example for a graph $G$ of a linear deterministic 3-user interference channel with $q = 4$ levels per user and channel gains: $n_{1,1} = 4$, $n_{2,2} = 4$, $n_{3,3} = 4$, $n_{1,2} = 3$, $n_{1,3} = 2$, $n_{2,1} = 2$, $n_{2,3} = 1$, $n_{3,1} = 3$ and $n_{3,2} = 2$

The selection of active and inactive levels translates to a *colouring* of vertices by two different colours. In order to decode interference-free signals, each pair of vertices, connected by an edge in a desired link, is always jointly selected as active or jointly selected as inactive. Further vertices, adjacent to these active vertices, must be selected as inactive to avoid interference. The edges contained in desired links describe only desired signals, all other edges are potential interference.

### B. A Constricted Maximum Independent Set Problem

The joint selection of a pair of vertices within a desired link is observed as an explicit redundancy. A straightforward reduction of this redundancy is an *edge-contraction* of all desired links within the graphical model. This idea has been used implicitly in [10] for a comparable problem. A formal edge-contraction of two vertices $v_1, v_2 \in V$ connected by a single edge $e = (v_1, v_2) \in E$ is a sequence of elementary operations on graphs [12]. For the resulting *contracted graph* $G' = (V', E')$, the edge-contraction of $e$ in $G$ is symbolically defined by the operator / (slash):

$$G' \quad := \quad G/e. \qquad (2)$$

Both vertices $v_1$ and $v_2$ are excluded from $V$ whereas a new (contracted) vertex $v'$ is included into $V$. The resulting vertex set $V'$ becomes

$$V' \quad = \quad (V - \{v_1, v_2\}) \cup \{v'\}. \qquad (3)$$

All the edges formerly incident to $v_1$ or $v_2$ are re-connected to the new $v' \in V'$. Accordingly, the resulting edge set becomes

$$E' \quad = \quad (E - \{(w, v) | w \in V, v \in \{v_1, v_2\}\}) \cup \{(w, v') | (w, v_1) \in E \cup (w, v_2) \in E, w \in V'\}. \qquad (4)$$

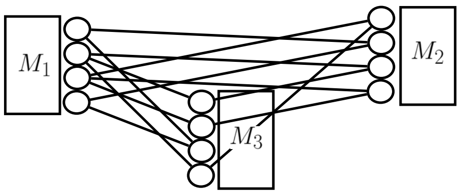


Fig. 2. An example for a *contracted graph* $G'$ of the linear deterministic 3-user interference channel as shown in Fig. 1

The edge-contraction is consecutively applied to each edge $e$ in the set of all desired links $E_{\mathsf{D}}$ to obtain the graph $G'$ with

$$E_{\mathsf{D}} := \bigcup_{i \in \mathcal{K}} \{(v_1, v_2) \in E | v_1 \in T_i, v_2 \in R_i\}, \quad (5)$$

$$G' = G/\{e \in E_D\}. \quad (6)$$

After such a contraction of all desired links, these links are compactly represented by a single vertex for each level. The remaining edges in the contracted graph solely describe potential interference. The new supernode, containing the set of contracted vertices from both sets $T_i$ and $R_i$, is denoted by $M_i$. Note that $|M_i| = q$ holds. Exemplarily, the contracted graph $G'$ of graph $G$ in Fig. 1 is depicted in Fig. 2.

We still consider the coding scheme as a colouring of vertices. On the contracted graph $G'$, the colouring describes a valid interference-free communication scheme, if the set of active vertices meets the condition of an independent set: An *independent set* $I$ is a subset of vertices, where no two different vertices $v, w \in I$ are adjacent to each other, i.e.:

$$I := \{S \subseteq V' | \forall v, w \in S : (v, w) \notin E'\}. \quad (7)$$

In general, such an independent set is ambiguous for graphs. Accordingly, the *family of independent sets* $\mathcal{I}$ is the set of all feasible independent sets of $G'$. In the following, the *set of locally active vertices* in link $M_k$ is denoted as $I_k := (I \cap M_k)$. This set is useful to define a normalized measure of communication theory. The *generalized degrees of freedom*, defined in [2], [7], describe the reduction of the transmission rate due to interference with regard to the capacity of the corresponding interference-free point-to-point link. In terms of independent sets, we interpret the GDoF as the independence ratio of link $k$ and define them as follows:

**Definition 2.** *An approximation of the* ***generalized degrees of freedom*** *per user* $k \in \mathcal{K}$ *is the ratio* $d_k$ *of the number of locally active vertices* $I_k$ *to the total number of vertices in* $M_k$*:*

$$d_k := \frac{|I_k|}{|M_k|}. \quad (8)$$

The main goal of a *fair* communication scheme can be expressed as the maximization of the minimal $d_k$ among each desired link $M_k$. I.e., the optimization problem is reduced to a *constricted maximum independent set problem* which desires to find an independent set $I$ in $\mathcal{I}$ so that the minimal GDoF $d_k$ of link $M_k$ are maximized:

$$\max_{I \in \mathcal{I}} \min_{k \in \mathcal{K}} d_k. \quad (9)$$

## III. The Greedy Algorithm

It is intended to find a maximized and almost symmetric distribution of the GDoF among all users. For the approximation of maximal independent sets in graphs, it is known from graph theory that good lower bounds on the cardinality of maximal independent sets are achieved by greedy algorithms with low computational complexity. Generally, greedy algorithms yield *locally optimal* but not necessarily globally optimal solutions. However, locally optimal solutions are useful to obtain distributed instead of centralized coding schemes.

The persecuted greedy approach intends to minimize the impact of interference caused by the activation of single vertices. To compare the number of affected vertices adjacent to an active vertex $v \in V'$, we use the measure of *edge-degree* $d(v)$, which is defined as the cardinality of its *neighbourhood* $\mathcal{N}(v)$:

$$\mathcal{N}(v) := \{w \in V' | (v, w) \in E'\}, \quad (10)$$

$$d(v) := |\mathcal{N}(v)|. \quad (11)$$

The *set of vertices with minimal edge-degree in node* $M_k$ is:

$$M_k^{\min} := \left\{ v \in M_k \middle| \min_{v \in M_k} d(v) \right\}. \quad (12)$$

**Algorithm 1** Min-Degree Greedy Coding Scheme

```
I_k ← ∅, ∀k;
while V' ≠ ∅ do
    for all k = 1, ..., K do
        v ← q ∈ M_k^min;
        I_k ← I_k ∪ v;
        for all w ∈ N(v) do
            for all x ∈ N(w) do
                E' ← E' − (w, x);
            end for;
            V' ← V' − w;
        end for;
        V' ← V' − v;
    end for;
end while;
return I_k, ∀k;
```

In the proposed coding scheme of Algorithm 1, all vertices are initially *unassigned* (neither as active, nor as inactive) and the sets $I_k$ are empty. In each iteration of the loop, the supernodes $M_k$ are scheduled round robin to assign a single vertex $v$ as active, arbitrarily selected from the current set $M_k^{\min}$. This selection is *greedy* for the minimal edge-degree. Vertex $v$ is included to the set of locally active vertices $I_k$ so that $d_k$ increases for link $k$. The vertices $w$ in the neighbourhood of vertex $v$ are implicitly set as inactive by removing them from the vertex set $V'$. All edges incident to these inactive vertices $w$ are also removed from the graph. Consequently, the edge-degrees of remaining vertices in $V'$, formerly adjacent to $v$, decrease. This reduction of edge-degrees supports the alignment of interference at these inactive vertices during successive iterations. In the last step of the current iteration, the selected vertex $v$ is finally removed from $V'$.

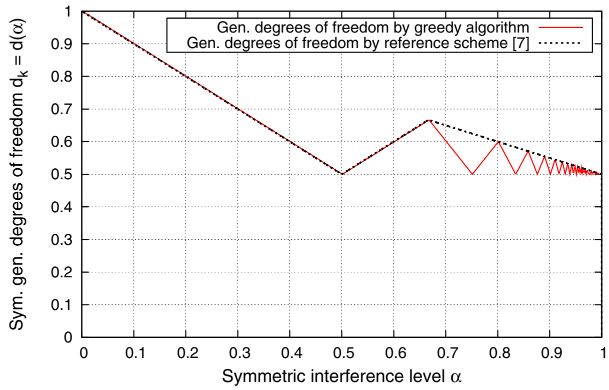


Fig. 3. Generalized degrees of freedom $d_k$ per link $M_k$ for the symmetric linear deterministic interference channel achieved by the greedy algorithm with $K = 3$ links, $q = 800$ levels and the interference level $\alpha \in (0, 1)$

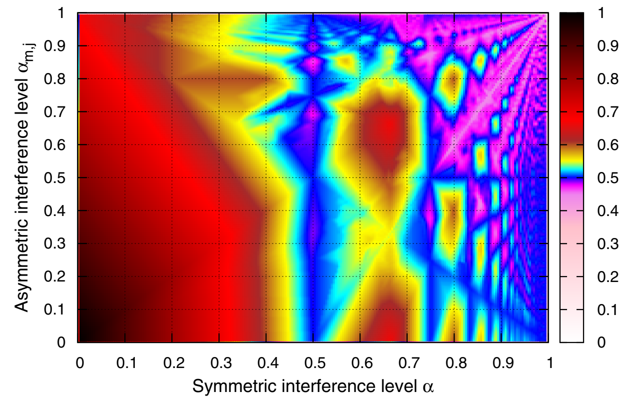


Fig. 4. Normalized achieved sum of the generalized degrees of freedom $\overline{d}(\alpha, \alpha_{m,j})$ for the symmetric interference level $\alpha \in (0, 1)$ and a single asymmetric interference level $\alpha_{m,j} \in (0, 1)$ with $m \neq j$ and $K = 3$ direct links, each link containing $q = 200$ levels

After all vertices in the graph have been assigned as active or inactive, the vertex set $V'$ is empty and the greedy algorithm terminates. The set $I_k$ is returned for each link $M_k$ and used as coding scheme.

## IV. Evaluation

In the following, approx. GDoF of the greedy algorithm are evaluated on two exemplary deterministic $K$-user interference channels having specific constraints on their channel gains.

### A. The Symmetric Deterministic Channel

We characterize the *symmetric* channel by these constraints:

- All the *direct channel gains* are equal, i.e. $n_{m,m} = q$, as given by the system model in Section II-A.
- Parameter $\alpha_{m,j}$ denotes the *interference level* [2] defined as the ratio between the number of interfered levels to the number of available signal levels at cross-link $(m, j)$:

$$\alpha_{m,j} \quad := \quad \frac{n_{m,j}}{q}. \qquad (13)$$

For symmetry, all *cross-channel gains* $n_{m,j}$ are restricted to equal interference levels $\alpha_{m,j} = \alpha \in (0, 1)$:

$$n_{m,j} = n_\mathsf{c} := \alpha q, \ \forall m \neq j, \ m, j \in \mathcal{K}. \qquad (14)$$

In Fig. 3, the *dashed black lines* represent the symmetric GDoF for the reference coding scheme, derived in [7, Theorem 3.1]. These GDoF are given as piecewise defined linear functions, depending on the interference level $\alpha$. Furthermore, note that *symmetric* GDoF imply that the $d_k$ are equal for all links $M_k$. The symmetric GDoF of our greedy algorithm are indicated by *solid red lines*. For $\alpha \in (0, \frac{2}{3}]$, this coding scheme is exactly congruent to the reference scheme. Apart from local maxima, the coding schemes differ within $\alpha \in (\frac{2}{3}, 1)$ due to omitting *interference cancellation* used in the reference scheme, i.e. we treat aligned interference as noise here.

**Proposition 1.** *The symmetric GDoF $d_k$ per link $M_k$, obtained by the greedy algorithm, are for a symmetric channel with interference level $\alpha$ and non-negative integers $n \in \mathbb{N}_0$:*

$$d(\alpha) = \begin{cases} -(n+1)\alpha + n + 1 & , \alpha \in [\kappa_1(n), \kappa_2(n)] \\ (n+1)\alpha - n & , \alpha \in [\kappa_2(n), \kappa_3(n)] \end{cases}. \qquad (15)$$

*The interval bounds $\kappa_1(n)$, $\kappa_2(n)$ and $\kappa_3(n)$ with $n \in \mathbb{N}_0$ are*

$$\kappa_1(n) := \frac{2n}{2n+1}, \ \kappa_2(n) := \frac{2n+1}{2n+2}, \ \kappa_3(n) := \frac{2n+2}{2n+3}. \quad (16)$$

A sketch of the proof is given in the appendix. In the first part of (15), each segment with a negative slope is characterized for the intervals $\alpha \in [\kappa_1(n), \kappa_2(n)]$. In the second part, the segments with a positive slope for the intervals $\alpha \in [\kappa_2(n), \kappa_3(n)]$ are described. Note $\kappa_3(n) = \kappa_1(n + 1)$.

### B. The Constrained Asymmetric Deterministic Channel

Based on the symmetric interference channel, a dedicated asymmetry is introduced now. The constraints of this channel are assumed to be as follows:

- The direct channel gains are equal: $n_{m,m} = q, \forall m \in \mathcal{K}$.
- The symmetric interference level $\alpha$ is defined according to Section IV-A, except that
- a single individual interference level $\alpha_{m,j} \in (0, 1)$ with an arbitrary $m \neq j$ is introduced as a unique asymmetry for exactly one interfering link $(m, j)$.

In contrast to the symmetric channel, the GDoF achieved by the greedy algorithm are not exactly symmetric any more. Nonetheless, we designed the greedy algorithm to approach almost symmetric GDoF. Thus, the *achieved sum of the generalized degrees of freedom*, normalized to the number of direct links, is considered, i.e., $\overline{d}(\alpha, \alpha_{m,j}) := \frac{1}{K} \sum_{k \in \mathcal{K}} d_k$.

The evaluation of the greedy algorithm reveals that $\overline{d}(\alpha, \alpha_{m,j})$ behaves as illustrated in Fig. 4. Note that the algorithm is repeated several times to approach the optimal allocation of levels for each link. For $\alpha_{m,j} = \alpha$, these results evidently reduce to those of the symmetric channel given in (15). For all other values of $\alpha_{m,j}$ and $\alpha$, the GDoF achieved by the greedy algorithm display well-structured symmetries of the considered channel. Clear local maxima occur at dedicated turning points similar to those in the symmetric case. We conjecture that these local maxima can not be improved by an additional interference cancellation approach as in the case of the symmetric channel. However, for the local minima a limited improvement of the GDoF is indeed possible.

## V. Conclusions

We studied the generalized degrees of freedom that are achieved by a greedy algorithm for linear deterministic $K$-user interference channels. To facilitate the communication problem, we merged all desired links in the corresponding graph representation. This idea has lead to a formulation of the optimization problem as a max-min independent set problem. We proposed a greedy algorithm using the interference alignment principle to approach the optimal solution for that problem. Known results for the symmetric $K$-user interference channel were achieved [7]. Furthermore, we showed that the greedy algorithm enables to investigate coding schemes on linear deterministic interference channels with asymmetric gains.

## Appendix

*Proof:* The input-output equation (1) can be simplified for the symmetric channel with respect to levels $l \in \mathbb{N}$ as

$$\mathbf{y}_j(l) = \mathbf{x}_j(l),\ 1 \leq l \leq q - n_c, \tag{17}$$

$$\mathbf{y}_j(l) = \sum_{m=1}^{K} \mathbf{x}_m(l - q + n_c),\ q - n_c < l \leq q. \tag{18}$$

Translated to graph $G'$, the $l$-th levels are only adjacent to the $l - q + n_c$-th and to the $l + q - n_c$-th levels of the other users. Hence, graph $G'$ is effectively decomposed into $q - n_{\mathsf{c}}$ subgraphs. All these subgraphs are mutually disjoint. Depending on the total number of levels $q$ per user and the cross-channel gain $n_{\mathsf{c}}$, the subgraphs contain either $l_1$ levels or $l_2$ levels, i.e., $l_1 := \lfloor q/q-n_{\mathsf{c}} \rfloor + 1$ and $l_2 := \lfloor q/q-n_{\mathsf{c}} \rfloor$. The number of subgraphs with $l_1$, $l_2$ levels is denoted by $c_1$, $c_2$, resp. (exhibiting similarities to *Turán graphs* [9, p. 207]), so that the levels are decomposed into a linear combination of $q$:

$$c_1 := q \mod (q - n_{\mathsf{c}}), \tag{19}$$

$$c_2 := (q - n_{\mathsf{c}}) - c_1, \tag{20}$$

$$\Rightarrow q - n_{\mathsf{c}} = c_1 + c_2,$$

$$\Rightarrow q = c_1 l_1 + c_2 l_2.$$

Each of these disjoint subgraphs is considered separately. A shift of $q - n_c$ levels in $G'$ is equivalent to a single shift in such a subgraph. We observe that these subgraphs are always isomorph to the graph depicted in Fig. 5, if restricted to $l = l_1$ or $l = l_2$ levels. The interference level $\alpha = \frac{l-1}{l}$ corresponds to the interval bounds $\kappa_1(n)$, $\kappa_2(n)$ and $\kappa_3(n)$ in (16).

The max-min independent sets of the subgraphs with an odd $l$ are achieved by alternately selecting all odd numbered levels as active (filled white) and all remaining levels as inactive (filled gray) as in [6]. The assignment of active/inactive levels in subgraphs with even $l > 2$ is also alternating aside from a single assignment of two successively inactive levels. By applying the greedy algorithm on such a subgraph, we obtain the same coding scheme. The GDoF achieved for an odd or even number of levels are given for integers $n \in \mathbb{N}_0$ resp.:

$$d(\kappa_1(n)) := 1 - \frac{\kappa_1(n)}{2} = \frac{n+1}{2n+1}, \tag{21}$$

$$d(\kappa_2(n)) := 1/2. \tag{22}$$

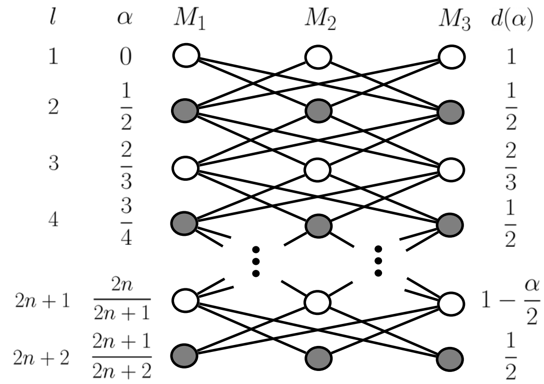


Fig. 5. A subgraph with $l$ levels of the contracted graph $G'$ for a symmetric channel with $K = 3$ links, interference level $\alpha = \frac{l-1}{l}$ and the corresponding generalized degrees of freedom $d(\alpha)$ achieved by the greedy algorithm

Due to the linear decomposition of $G'$ into these two classes of subgraphs, the GDoF for the first part of (15) are simply linear functions from points $(\kappa_1(n), d(\kappa_1(n)))$ to points $(\kappa_2(n), d(\kappa_2(n)))$, so that the GDoF are

$$d(\alpha) = -(n+1)\alpha + n + 1 \tag{23}$$

within the separate intervals $\alpha \in [\kappa_1(n), \kappa_2(n)]$. The second part of (15) is derived analogously for $\alpha \in [\kappa_2(n), \kappa_3(n)]$. ■

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