Uniform and Non-Uniform Delay-Rate Tradeoffs in Partial Ergodic Interference Alignment

Henning Maier and Rudolf Mathar

Institute for Theoretical Information Technology
RWTH Aachen University, 52056 Aachen, Germany
Email: {maier, mathar}@ti.rwth-aachen.de

***Abstract*—The impact of significantly long delays is a central problem one has to face in the ergodic interference alignment scheme. In this work, we consider the delay-rate tradeoff in a fully connected K-user Gaussian interference channel through the lens of partial interference alignment. In the concept of partial interference alignment, only a subset of user-pairs with a favorable interference pattern is aligned. We explore two flexible communication schemes: At first, we propose a scheme with uniform delay-rate demands. Then we introduce the delay-rate tradeoff for a case of non-uniform demands such that also imbalanced delays and rates can be investigated.**

I. Introduction

The principle of *interference alignment* (IA) [1] is a novel technique to achieve extraordinary high data rates in multi-user interference channels. In particular, the sum-capacity of known IA schemes are shown to achieve half of the interference-free capacity for each communicating user-pair, independent of the total number of users.

Among a diversity of theoretically achievable IA schemes, the basic idea of *ergodic* IA [2]–[4] is to exploit pairings of only two channel matrices that are well chosen over time s.t. multi-user interference of the effective channel can be completely cancelled out by the means of a repetition code. Assuming that the channel coefficients are quantized, independently distributed and ergodic over time, the desired pair of matrices does indeed occur with a positive probability. Since two specific channel realizations already suffice for the interference-free transmission of K symbols, the conjectured alignment is achievable, even at a finite signal-to-noise ratio [5].

However, such an ergodic IA scheme admits the decoding delay and storage space at the transmitters and receivers to scale significantly. This is because the receivers can not decode and complete the ergodic IA scheme until the proper second channel matrix of the desired pairing has occurred. Ergodic IA requires around $\mathcal{O}(c^{K^2})$ time steps for a (quantization) constant c and a number of K user-pairs [6]. An implementation appears impractial yet. Nonetheless, this is still a very promising information-theoretic approach.

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In order to reveal more practical solutions, a relaxation of the restrictive conditions of ergodic IA is investigated in [6], [7] in terms of a *delay-rate tradeoff* (DRT). In [7], the DRT is achieved by extending the repetition code over more than only two channel instances. Therein, IA is achieved if an *alignment set*, i.e., the sum of a set of more than two channel realizations, results in an effectively interference-free channel. In the approach pursued in [6], it suffices that interference can also be removed by finding complementary sets in submatrices of the channel instead. In both studies, the main idea is to augment the set of admissible matrices to reduce the delay albeit at the expense of maximal sum-rate.

Performing only *partial* IA as, e.g., in [8], means to align only a subset of user-pairs with a favorable interference pattern. The remaining unaligned users discard their received signals and admit interference and hence a rate-loss.

Contributions. In the present paper, we combine the ideas of partial and ergodic interference alignment and investigate the resulting DRT. As in most current IA schemes, we firstly assume uniform delay and rate demands among all users. We then extend our partial IA scheme to non-uniform delay and rate demands for one user-pair. Our methodical tools mainly rely on probability theory and combinatorics [9].

Organization. The system model and a polar quantization scheme is provided in Section II. We briefly review the ergodic IA scheme in Section III. *Partial* ergodic IA is introduced in Section IV. The resulting DRT is investigated w.r.t. a uniform parameterization in IV-A, and w.r.t. a more general non-uniform parameterization in Section IV-B. We discuss some essential differences of the schemes provided here to those of [6], [7] in Section V and conclude in Section VI.

II. System Model

We consider the Gaussian wireless channel model as also proposed by Nazer et al. in [3]. In the given scenario, there is a fully-connected K-user interference channel with K transmitting source nodes Tx_k and K receiving destination nodes Rx_k with pairwise indexed users $k \in \mathcal{K} = \{1, \dots, K\}$. Each source node Tx_k desires to transmit a uniform i.i.d. message $w_k \in \mathcal{W} = \{1, 2, \dots, 2^{n\widetilde{R}_k}\}$ to its dedicated destination node Rx_k. A transmitter Tx_k encodes its message w_k into the complex-valued input signal $\{X_k(t)\}_{t=1}^{T}$ with rate $\widetilde{R}_k \geq 0$

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and block length T using an encoding function $\mathcal{E}_k : \mathcal{W} \to \mathbb{C}^T$. The input symbols must satisfy the average power constraint $\frac{1}{T}\sum_{t=1}^{T} |X_k(t)|^2 \le P$. A receiver Rx_k uses the received output signal $\{Y_k(t)\}_{t=1}^{T}$ to decode its received message with a decoding function $\mathcal{D}_k : \mathbb{C}^T \to \mathcal{W}$ and estimates a message $\widehat{w}_k$.

The wireless channel is subject to fast Rayleigh fading, modeled by a $K \times K$ matrix $\mathbf{H}(t) = \{h_{kl}(t)\}_{k,l}$ of time-varying channel coefficients. Rows are indexed by k and columns by l. The set of all possible channel matrices is denoted by $\mathcal{H}$. Each complex channel coefficient is i.i.d. by $h_{kl}(t) \sim \mathcal{CN}(0,1)$ in discrete time t and space. This implies that the phase-shifts φ are uniformly i.i.d. in $[0, 2\pi)$ and the magnitudes are independently Rayleigh distributed. The additive Gaussian noise is i.i.d. with $Z_k(t) \sim \mathcal{CN}(0,1)$ at a receiver k. The input-output relationship of signals in the complex baseband-equivalent model at receiver k is determined by the following linear combination:

$$Y_k(t) = \sum\nolimits_{l\in\mathcal{K}} h_{kl}(t)X_l(t) + Z_k(t), \quad \forall k \in \mathcal{K}. \tag{1}$$

A. Polar Quantization Scheme

We propose a quantization scheme slightly different to the one given in [3]. A *polar quantization* [10], [11] is performed to obtain a finite set of channel matrices $\widehat{\mathcal{H}}$ from $\mathcal{H}$ enabling ergodic IA. The quantization function $q : \mathcal{H} \to \widehat{\mathcal{H}}$ maps the complex coefficients in $\mathcal{H}$ to corresponding quantization points in $\widehat{\mathcal{H}}$, i.e., the centroids of the closest quantization cell. The cells are described by a segmentation of κ concentric rings with $\eta_i = \eta$ sectors of equal angle $\frac{2\pi}{\eta_i}$ per ring as depicted in Figure 1. The quantized coefficients are limited to a sufficiently large magnitude $r_\kappa = h_{\max}$. All coefficients larger than $h_{\max}$ are mapped to a single quantization point at infinity.

In contrast to [3], the concentric rings are *not* chosen to have an equal width, i.e., $w_i := r_i - r_{i-1} \ne w$. Instead, we assume that there exists a proper choice of radii s.t. each quantization cell occurs with an approximately uniform probability. If κ and η are sufficiently large, an arbitrarily small quantization error can be achieved. This approach can considerably simplify our reasoning in comparison to the case of equal widths. Hence, we can assume a total number of $|\widehat{\mathcal{H}}| = (1+\eta\kappa)^{K^2}$ uniformly i.i.d. quantized channel realizations. Non-uniform and i.i.d. coefficients would severely complicate our analysis. We assume that perfect causal channel state information (CSI) of all links is globally available.

III. Ergodic Interference Alignment

An ergodic rate tuple $(R_1, R_2, \ldots, R_k)$ is *achievable* if, for $\epsilon > 0$ and n large enough, there exists channel code with $\mathcal{E}_1, \ldots, \mathcal{E}_K, \mathcal{D}_1, \ldots, \mathcal{D}_K$ s.t. $\widetilde{R}_k > R_k - \epsilon$, $k \in \mathcal{K}$, and error probability $\Pr\left(\bigcup_{k\in\mathcal{K}}\{\widehat{w}_k \ne w_k\}\right) < \epsilon$ holds. In [2], Nazer et al. proved that an ergodic rate of $R_k \le \frac{1}{2}\mathsf{E}[\log(1 + 2|h_{kk}|^2 P)]$ for all $k \in \mathcal{K}$ is achievable by an ergodic IA scheme.

Basically, the interference-free transmission of a single symbol is performed over two carefully chosen time-steps t_1 and t_2 with $t_1 < t_2$. In the *primary step* t_1, a new set of K symbols $X_k(t_1)$ is transmitted over $\widehat{\mathbf{H}}(t_1)$. The received signals are each interfered by all undesired $K-1$ signals and only stored but not decoded at the receivers yet.

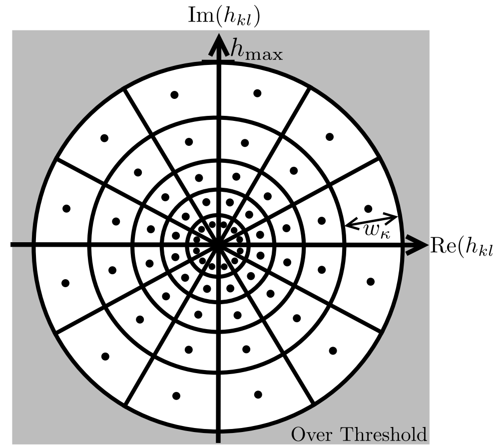

Fig. 1. The polar quantization function $q : \mathcal{H} \to \widehat{\mathcal{H}}$ maps complex channel coefficients in $\mathcal{H}$ to the centroids of quantization cells in $\widehat{\mathcal{H}}$ with $\kappa = 5$ concentric rings and $\eta_i = \eta = 12$ sectors of equal angle $\frac{2\pi}{\eta}$ s.t. each quantization point is uniformly i.i.d. The widths w_i of the concentric rings are not equal.

The following *complementarity condition* defines a pairing of two specific realizations of the quantized channel matrices at some later time-step t_2. Their sum is effectively interference-free and yields a scaled identity matrix:

$$\widehat{\mathbf{H}}(t_1) + \widehat{\mathbf{H}}^{\mathsf{C}}(t_2) = 2\mathbf{I}, \quad t_1 < t_2. \tag{2}$$

In other words, it is sufficient that each entry of the matrix $\widehat{\mathbf{H}}^{\mathsf{C}}(t_2)$ is the additive inverse of the corresponding entry in $2\mathbf{I} - \widehat{\mathbf{H}}(t_1)$. The cases where the magnitude of a channel coefficient is beyond $h_{\max}$ are excluded from IA.

Given that condition (2) holds for the eventually significant later *secondary step* t_2, the interference perceived in step t_1 is completely cancelled out by a retransmission of all K signals $X_k(t_2) = X_k(t_1)$ over $\widehat{\mathbf{H}}^{\mathsf{C}}(t_2)$. Matrices that are not complementary to $\widehat{\mathbf{H}}(t_1)$ are used for other pairings.

The *expected sum-rate* $\mathsf{E}[\widetilde{R}_\Sigma]$ of K aligned users can be expressed by the rate $\widetilde{R} := \widetilde{R}_k > \mathsf{E}[\log(1 + 2|h_{kk}|^2 P)] - \epsilon$ of an interference-free point-to-point channel as follows:

$$\mathsf{E}[\widetilde{R}_\Sigma] = \mathsf{E}\left[\sum\nolimits_{k\in\mathcal{K}} \widetilde{R}_k\right] = \frac{K}{2}\widetilde{R}. \tag{3}$$

This strategy demands very high delays and a vast storage space as each pair of t_1 and t_2 may be far apart. The *delay* is defined as their absolute time difference $D = t_2 - t_1$. Additional propagation delay is neglected. Since the quantized coefficients are given as uniformly i.i.d., we can denote the *success probability* that the required complementary matrix $\mathbf{H}^{\mathsf{C}}(t_2)$ is encountered in the secondary step t_2 as:

$$p_0 := \Pr[\widehat{\mathbf{H}}(t) = \widehat{\mathbf{H}}^{\mathsf{C}}(t_2)] = |\widehat{\mathcal{H}}|^{-1} = (1+\eta\kappa)^{-K^2}. \tag{4}$$

The *expected delay* of a successfully aligned transmission yields from a geometrically distributed random variable [7], [9]:

$$\mathsf{E}[D] = p_0^{-1} = |\widehat{\mathcal{H}}| = (1+\eta\kappa)^{K^2}. \tag{5}$$

We denote $\log_{1+\kappa\eta}(\mathsf{E}[D])$ as the *expected delay exponent* [6].

IV. Partial Ergodic Interference Alignment

In analogy to the studies of the DRT in [6], [7], we also intend to relax the restrictive complementarity condition (2) in order to shorten the expected delay at the expense of maximal sum-rate. We can assume w.l.o.g. that there always exists a matrix $\widetilde{\mathbf{H}}^{\mathsf{C}}(t_2) \in \widehat{\mathcal{H}}$ with $\widetilde{\mathbf{H}}^{\mathsf{C}}(t_2) \neq \widehat{\mathbf{H}}^{\mathsf{C}}(t_2)$, which may still be added to $\widehat{\mathbf{H}}(t_1)$. Then, an *error matrix* $\mathbf{E}(t_2) = \{e_{ij}(t_2)\}_{i,j}$ must be added to the scaled identity matrix in order to compensate the missing complementarity:

$$\widehat{\mathbf{H}}(t_1) + \widetilde{\mathbf{H}}^{\mathsf{C}}(t_2) = 2\mathbf{I} + \mathbf{E}(t_2). \tag{6}$$

If $\mathbf{E}(t_2) = \mathbf{0}$, (*exact*) complementarity, $\widetilde{\mathbf{H}}^{\mathsf{C}}(t_2) = \widehat{\mathbf{H}}^{\mathsf{C}}(t_2)$, holds. Here, a sparse $\mathbf{E}(t_2)$ is sufficient for *partial* ergodic IA.

A matrix entry $e_{ij}(t_2) \neq 0$ implies that the received signal $Y_i(t_2)$ would contain interference from transmitter j if repetition coding were performed at time t_2. Then, receiver i cannot decode its desired signal and there is an *erasure* at receiver i. We call the event that a received signal at a receiver i is erased and not decoded, an *alignment dropout* (ADO). As CSI is globally available, all users are aware of such an ADO.

Dropped signals cause a rate-loss and may undergo retransmission or an elaborate error correction protocol. But for the sake of simplicity, further measures to correct or decode these erased symbols, e.g., decoding strong interference or treating interference as noise, are neglected.

Since a dropped user k discards the entire received signal $Y_k(t_2)$, all entries $e_{kl}(t_2)$ in row k may be non-zero, i.e., the interference is enabled to align within the full signal space of Rx_k at t_2. But the interference caused by transmitter Tx_k remains aligned at all receivers Rx_l where $e_{kl}(t_2) = 0$ holds.

We adapt the terminology of [7], [8] to our partial ergodic IA scheme. The *partial alignment set* for a given matrix of the primary step $\widehat{\mathbf{H}}(t_1)$ under the relaxed condition (6) is denoted by the set of all matrices:

$$\mathcal{A}_{\widehat{\mathbf{H}}(t_1)} := \left\{ \widetilde{\mathbf{H}}^{\mathsf{C}}(t_2) \in \widehat{\mathcal{H}} \right\}, \tag{7}$$

that are valid under some explicitly defined conditions, e.g., the (partial) alignment set of the original ergodic IA scheme may only contain the complementary matrix $\widehat{\mathbf{H}}^{\mathsf{C}}(t_2)$:

$$\left\{ \mathcal{A}_{\widehat{\mathbf{H}}(t_1)} : \widetilde{\mathbf{H}}^{\mathsf{C}}(t_2) = \widehat{\mathbf{H}}^{\mathsf{C}}(t_2) \right\} \Leftrightarrow \left\{ \mathcal{A}_{\widehat{\mathbf{H}}(t_1)} : \mathbf{E}(t_2) = \mathbf{0} \right\}. \tag{8}$$

The definition of a partial alignment set implies the number of all matrices admissible for the secondary step t_2 and hence determines the expected delay $\mathsf{E}[D]$ and the expected sum-rate $\mathsf{E}[\widetilde{R}_\Sigma]$ which characterize the DRT.

Note that the alignment set defined in [8] addresses the indices of aligned users instead of the channel matrices valid for IA as given in [7] and in the present paper.

A. The Uniform Delay-Rate Tradeoff

A very basic scheme for partial ergodic IA is to globally restrict the number of ADOs uniformly among all users. Thus, if the error matrix $\mathbf{E}(t_2)$ in the partial alignment set is relaxed to non-zero entries in $0 \leq J < K$ rows, only a maximal number of up to J receivers may be dropped.

More formally, let $L(t)$ denote the random number of ADOs at time t and let the indices of currently dropped users be in a set $\mathcal{L}(t)$ s.t. the cardinality of this set is $|\mathcal{L}(t)| = L(t)$. Parameter J decides if the current realization of the channel matrix is either used for a primary step (if $L(t) > J$ holds) or a secondary step (if $L(t) \leq J$ holds). Altogether, the partial alignment set of (7) is constrained by a *global decision threshold* of *exactly* $L(t_2) = J$ ADOs:

$$\mathcal{A}_{\widehat{\mathbf{H}}(t_1)}(J) := \left\{ \mathcal{A}_{\widehat{\mathbf{H}}(t_1)} : L(t_2) = J \right\}. \tag{9}$$

The number of admissible channel matrices satisfying this alignment set clearly grows with increasing J. We compute the probability that there are $i < K$ or less ADOs at time t as:

$$\begin{aligned} p_i &:= \Pr[\widehat{\mathbf{H}}(t) \in \bigcup\nolimits_{j=0}^{i} \mathcal{A}_{\widehat{\mathbf{H}}(t_1)}(j)] \\ &= \Pr[L(t) \leq i] = (1 + \eta\kappa)^{-(K-i)K}. \end{aligned} \tag{10}$$

We fix $p_{-1} = 0$ for notational convenience. A matrix in the alignment set with exactly $L(t) = i$ ADOs occurs with probability:

$$\Pr[\widehat{\mathbf{H}}(t) \in \mathcal{A}_{\widehat{\mathbf{H}}(t_1)}(i)] = \Pr[L(t) = i] = p_i - p_{i-1}. \tag{11}$$

Summing up the probabilities of (11) for $i = 0, \ldots, J$ yields a telescoping series, i.e., $\sum_{i=0}^{J}(p_i - p_{i-1}) = p_J$, and hence (10) with $i = J$.

All K pairs are dropped with probability:

$$\Pr[L(t) = K] = 1 - \Pr[L(t) \leq K-1] = 1 - p_{K-1}. \tag{12}$$

The expected delay of a successful transmission with maximal J ADOs corresponds to:

$$\mathsf{E}[D] = \Pr[L(t) \leq J]^{-1} = p_J^{-1}. \tag{13}$$

The expected rate achieved by a single user-pair k is:

$$\begin{aligned} \mathsf{E}[\widetilde{R}_k] &= \Pr[k \notin \mathcal{L}(t) | L(t) \leq J] \frac{\widetilde{R}}{2} \\ &= \sum\nolimits_{j=0}^{J} \frac{p_j - p_{j-1}}{p_J} \frac{\binom{K-1}{j}}{\binom{K}{j}} \frac{\widetilde{R}}{2} \\ &= \sum\nolimits_{j=0}^{J} \frac{p_j - p_{j-1}}{p_J} \frac{K-j}{K} \frac{\widetilde{R}}{2} \\ &= \left[K - J + \sum\nolimits_{j=0}^{J-1} \frac{p_j}{p_J} \right] \frac{\widetilde{R}}{K2}. \end{aligned} \tag{14}$$

The expression (14) is explained as follows: In this scheme, partial IA always occurs whenever $L(t) \leq J$, independent of any pair k. As the rate only includes those cases where partial IA is successful, we must condition on the event that $L(t) \leq J$. However, a single user-pair can achieve rate $\widetilde{R}/2$ only if it is not dropped, i.e., $k \notin \mathcal{L}(t)$. For a number of exactly j ADOs, the pair k is not among the dropped pairs in $\binom{K-1}{j}$ of $\binom{K}{j}$ cases. Otherwise, if user-pair k is dropped, its rate is zero.

Accordingly, the expected sum-rate is easily computed:

$$\mathsf{E}[\widetilde{R}_\Sigma] = \sum\nolimits_{k=1}^{K} \mathsf{E}[\widetilde{R}_k] = \left[K - J + \sum\nolimits_{j=0}^{J-1} \frac{p_j}{p_J} \right] \frac{\widetilde{R}}{2}. \tag{15}$$

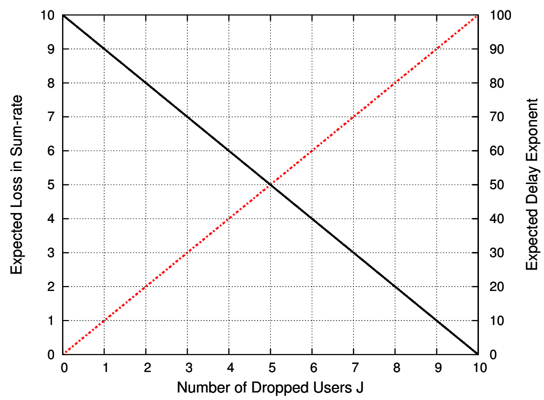

Fig. 2. The uniform DRT of partial ergodic IA is depicted for $J = 1, \ldots, K$ dropped users of $K = 10$ users in total. The rate $\widetilde{R}$ is normalized to $\widetilde{R} = 2$. The dotted red curve denotes the loss in the expected sum-rate, $\widetilde{R}\frac{K}{2} - \mathsf{E}[\widetilde{R}_\Sigma]$. The solid black curve denotes the expected delay exponent, $\log_{1+\eta\kappa}(\mathsf{E}[D])$.

We observe that $\mathsf{E}[\widetilde{R}_\Sigma] \approx (K-J)\frac{\widetilde{R}}{2}$ if the term $\sum_{i=0}^{J-1} \frac{p_i}{p_J}$ is neglected[1] w.r.t. $K - J$. This is not surprising since the channel is reduced to $K - J$ pairs in the worst case.

The expected *loss* in sum-rate w.r.t. conventional ergodic IA is denoted by $\widetilde{R}\frac{K}{2} - \mathsf{E}[\widetilde{R}_\Sigma]$ and increases approximately linear with parameter J. The expected delay exponent decreases linearly with J, i.e., $\log_{1+\eta\kappa}(\mathsf{E}[D]) = K(K-J)$. The delays and rates are illustrated in Fig. 2 for different values of the global decision threshold J.

B. On the Non-Uniform Delay-Rate Tradeoff

In the formerly introduced uniform partial IA scheme, each user experiences the same expected delay and rate depending on the single parameter J, at least in an ergodic sense. However, the delay and rate demands in practical systems may be heterogeneous and time-varying. Therefore, the communication scheme should be flexible by allowing an individual adjustment of the parameter J for each user-pair and at each time-step. With the following approach, it is possible to avoid (or to intentionally induce) extraordinarily short or long delays for a single dedicated user in a tradeoff.

Our suggested approach is to extend the uniform scheme. A simple method to prioritize or penalize specific user-pairs through a partial IA scheme is to allocate an individual number of admissible ADOs J_k per user-pair k whenever user-pair k belongs to the currently aligned users. Let vector $\mathbf{j} = (J_1, \ldots, J_K)$ describe such an allocation. This vector is a generalization of the parameter J in Section IV-A, where all J_k were equally (uniformly) chosen as $J_1 = \ldots = J_K = J$. An intuitive property of this parameter is that a user-pair k with a high valued J_k should be aligned with a higher probability than another pair $k' \neq k$ with a lower $J_{k'} < J_k$. Note that only the parameterization $\mathbf{j}$ is *non-uniform* here. The quantized coefficients remain uniformly i.i.d. as given in Section II.

For the users that are not aligned at time t, the entries in vector $\mathbf{j}$ are temporarily set to zero for the current realization.

[1] E. g, for sufficiently large parameters like $\eta = 12$, $\kappa = 10$ and $K = 10$, the term $\sum_{i=0}^{J-1} \frac{p_i}{p_J}$ is clearly negligible.

The time-dependent vector $\mathbf{j}(t)$ is updated at each time-step t according to the current error matrix $\mathbf{E}(t)$:

$$\mathbf{j}(t) := \begin{cases} J_k(t) = 0, & \text{if } \exists l \in \mathcal{K}, l \neq k : e_{k,l}(t) \neq 0, \\ J_k(t) = J_k, & \text{if } \forall l \in \mathcal{K}, l \neq k : e_{k,l}(t) = 0. \end{cases} \tag{16}$$

The time-variant *global decision threshold* of this non-uniform IA scheme is defined as the maximal J_k:

$$J^*(t) := \max_{k \in \mathcal{K}} J_k(t). \tag{17}$$

The partial alignment set with the non-uniform parameter $\mathbf{j}(t_2)$ can now be expressed by the given threshold as:

$$\mathcal{A}_{\mathbf{H}(t_1)}(J^*(t_2)) := \left\{ \mathcal{A}_{\mathbf{H}(t_1)} : L(t_2) \leq J^*(t_2) \right\}. \tag{18}$$

We confine our investigation to the case where almost all J_k are equal to J except for a single pair, s.t. an allocation $\mathbf{j} = (J_1 = J^*, J_2 = J, \ldots, J_K = J)$ with $J^* > J$ may be assumed w.l.o.g. The probabilities that $J_k(t)$ is either zero or J_k are mutually independent for all $k \in \mathcal{K}$:

$$\begin{aligned} \Pr[J_k(t) = J_k] &= \Pr[k \notin \mathcal{L}(t)] \\ &= \sum\nolimits_{j=0}^{K-1} (p_j - p_{j-1}) \frac{K-j}{K}, \end{aligned} \tag{19}$$

$$\Pr[J_k(t) = 0] = \Pr[k \in \mathcal{L}(t)] = 1 - \Pr[k \notin \mathcal{L}(t)]. \tag{20}$$

If a user-pair k is not dropped, $J_k(t) = J_k$ holds. The global decision threshold $J^*(t)$ of the alignment set can attain one of the three values J^*, J or 0 at each step t:

$$\Pr[J^*(t) = J^*] = \Pr[J_1(t) = J^*] = \Pr[1 \notin \mathcal{L}(t)], \tag{21}$$

$$\Pr[J^*(t) = 0] = \Pr[L(t) = K], \tag{22}$$

$$\Pr[J^*(t) = J] = 1 - \Pr[J^*(t) = J^*] - \Pr[J^*(t) = 0]. \tag{23}$$

If the first pair is not dropped, the global threshold is maximal with the value J^* due to the given parameterization. The global threshold is only zero if all K users are dropped. In the remaining cases, the threshold must be J since at least one user-pair k except the first is not dropped.

Here, partial ergodic IA is always successful if the number of ADOs is below the current global threshold:

$$\begin{aligned} p_{\text{success}} &:= \Pr[L(t) \leq J^*(t)] \\ &= \Pr[0 \leq L(t) \leq J] + \Pr[1 \notin \mathcal{L}(t), J < L(t) \leq J^*] \\ &= p_J + \sum\nolimits_{j=J+1}^{J^*} (p_j - p_{j-1}) \frac{K-j}{K}, \end{aligned} \tag{24}$$

i.e., we can reapply the ordinary uniform threshold and include the special case when the first pair is not dropped.

We commence the analysis of the DRT for the first pair. For this pair, the second step of partial ergodic IA can be completed if at most J^* users are dropped except the first pair. The expected delay of a successful transmission from the first user-pair results from the geometric distribution:

$$\begin{aligned} \mathsf{E}[D_1] &= \Pr[1 \notin \mathcal{L}(t), 0 \leq L(t) \leq J^*]^{-1} \\ &= \left[\sum\nolimits_{j=0}^{J^*} (p_j - p_{j-1}) \frac{K-j}{K} \right]^{-1}. \end{aligned} \tag{25}$$

The rate of the first user-pair is either $\widetilde{R}/2$ or zero for parameters $J_1(t) = J^*$ or $J_1(t) = 0$, respectively. Thus, the expected rate of the first user-pair yields:

$$\begin{aligned}\mathsf{E}[\widetilde{R}_1] &= \Pr[1 \notin \mathcal{L}(t) | 0 \leq L(t) \leq J^*(t)] \frac{\widetilde{R}}{2} \\ &= \sum\nolimits_{j=0}^{J^*} \frac{p_j - p_{j-1}}{p_{\text{success}}} \frac{K-j}{K} \frac{\widetilde{R}}{2}. \end{aligned} \tag{26}$$

For the remaining user-pairs with indices $k = 2, \ldots, K$, there are two cases to discern: On the one hand, a user-pair $k \neq 1$ enables partial ergodic IA only if $J^*(t_2) = J_k(t_2) = J$ holds, i.e., the first pair is dropped but pair k is not dropped, and $1 \leq L(t_2) \leq J$ holds:

$$\begin{aligned}&\Pr[1 \in \mathcal{L}(t), k \notin \mathcal{L}(t), 1 \leq L(t) \leq J] \\ &= \sum\nolimits_{j=1}^{J} (p_j - p_{j-1}) \frac{\binom{K-2}{j-1}}{\binom{K-1}{j-1}} \\ &= \sum\nolimits_{j=1}^{J} (p_j - p_{j-1}) \frac{K-j-1}{K-1}. \end{aligned} \tag{27}$$

On the other hand, a user-pair $k = 2, \ldots, K$ can be aligned alongside the first user-pair if both the first and the considered pair k are not dropped among in up to J^* ADOs:

$$\begin{aligned}&\Pr[1 \notin \mathcal{L}(t), k \notin \mathcal{L}(t), 0 \leq L(t) \leq J^*] \\ &= \sum\nolimits_{j=0}^{J^*} (p_j - p_{j-1}) \frac{\binom{K-2}{j}}{\binom{K}{j}} \\ &= \sum\nolimits_{j=0}^{J^*} (p_j - p_{j-1}) \frac{(K-j)(K-j-1)}{K(K-1)}. \end{aligned} \tag{28}$$

These two cases exclude each other. The expected delay of a successful transmission from a user-pair $k = 2, \ldots, K$ yields from the geometric distribution:

$$\begin{aligned}\mathsf{E}[D_k] = (&\Pr[1 \in \mathcal{L}(t), k \notin \mathcal{L}(t), 1 \leq L(t) \leq J] + \\ &\Pr[1 \notin \mathcal{L}(t), k \notin \mathcal{L}(t), 0 \leq L(t) \leq J^*])^{-1}. \end{aligned} \tag{29}$$

The expected rate of a user-pair $k = 2, \ldots, K$ is:

$$\begin{aligned}\mathsf{E}[\widetilde{R}_k] &= [\Pr[1 \in \mathcal{L}(t), k \notin \mathcal{L}(t) | L(t) \leq J^*(t)] + \\ &\quad \Pr[1 \notin \mathcal{L}(t), k \notin \mathcal{L}(t) | L(t) \leq J^*(t)]] \frac{\widetilde{R}}{2} \\ &= \left[\frac{\Pr[1 \notin \mathcal{L}(t), k \notin \mathcal{L}(t), 0 \leq L(t) \leq J^*]}{p_{\text{success}}} + \right. \\ &\quad \left. \frac{\Pr[1 \in \mathcal{L}(t), k \notin \mathcal{L}(t), 1 \leq L(t) \leq J]}{p_{\text{success}}} \right] \frac{\widetilde{R}}{2}. \end{aligned} \tag{30}$$

Altogether, the expected sum-rate of all K users is:

$$\mathsf{E}[\widetilde{R}_\Sigma] = \mathsf{E}[\widetilde{R}_1] + (K-1)\mathsf{E}[\widetilde{R}_k], \quad k \neq 1. \tag{31}$$

V. Discussion

In both of our presented schemes, the users wait for specific channel realizations to reduce the delay of completing ergodic IA for all K user-pairs. We observe that the proposed error matrix could be a helpful tool for the inclusion of some conventional error correcting protocols.

The given partial IA schemes could also be extended to more than two time-steps. Such an approach would be similar to the schemes in [6], [7]. A main advantage of our partial IA scheme is that the decoding of a subset of symbols can be performed without waiting for complete complementarity of a chain of matrices as it was done in the former schemes.

Furthermore, an adjustment of individual delay-rate demands can be set flexibly with parameter $\mathbf{j}$. The parameters can be changed in each time step without sacrificing the stored signals and hence causing no further rate losses. It is certainly possible albeit cumbersome to adapt the schemes to channels with a non-uniform distribution of channel gains.

A problem that arises in the schemes in [6], [7] concerns the accuracy in the estimations of CSI. The assumption of inaccurate CSI can severely affect the alignment task, when adding up many channel instances of several time steps. In our scheme, the remaining error would only concern the error of a single addition.

We did not fully solve the general non-uniform DRT for partial ergodic IA yet. A more general consideration involves arbitrary numbers of user-pairs allocating the same J_k.

VI. Conclusions

We investigated the delay-rate tradeoff for two partial ergodic Interference Alignment schemes. Our first scheme assumes a uniform choice of parameters for all users. Our second scheme includes a simplified non-uniform choice of parameters. The probabilistic analysis of the given schemes relies on the assumption of uniformly distributed channel coefficients after our proposed non-uniform polar quantization function has been applied. We provide an evaluation of the expected delays and expected sum-rates characterizing the delay-rate tradeoff.

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