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Exercise 1

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Problem 1. *(Properties of Positive Definite and Non-negative Definite Matrices)* Let $\mathbf{M} \in \mathbb{R}^{n\times n}$ be a matrix. Prove the following statements.

a) If $\mathbf{M}$ is non-negative definite, then for every vector $\mathbf{x} \in \mathbb{R}^n$ it holds that $\mathbf{x}^T\mathbf{M}\mathbf{x} \geq 0$.

b) If $\mathbf{M}$ is positive definite, then for every vector $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$, it holds that $\mathbf{x}^T\mathbf{M}\mathbf{x} > 0$.

Problem 2. *(Matrix Loewner Ordering Properties)* Let $\mathbf{V}$ and $\mathbf{W}$ be two $n \times n$ non-negative definite matrices, such that $\mathbf{V} = (v_{ij}) \preceq \mathbf{W} = (w_{ij})$, with the eignevalues as:

- $\lambda_1(\mathbf{V}) \geq \cdots \geq \lambda_n(\mathbf{V})$,
- $\lambda_1(\mathbf{W}) \geq \cdots \geq \lambda_n(\mathbf{W})$

Prove the following statements.

a) $\lambda_i(\mathbf{V}) \leq \lambda_i(\mathbf{W})$, for $i = 1, \dots, n$

b) $v_{ii} \leq w_{ii}$, for $i = 1, \dots, n$

c) $v_{ii} + v_{jj} - 2v_{ij} \leq w_{ii} + w_{jj} - 2w_{ij}$

d) $\mathrm{tr}(\mathbf{V}) \leq \mathrm{tr}(\mathbf{W})$

e) $\det(\mathbf{V}) \leq \det(\mathbf{W})$

Problem 3. *(Properties of Spectral Norm)* Spectral norm of a symmetric matrix $\mathbf{M}$ is given by $\rho(\mathbf{M}) = \max\limits_{1\leq i\leq n} |\lambda_m|$.

a) Show that $\rho(\mathbf{M})$ is not a norm for non-symmetric matrices.
(Hint: show that $\rho(\mathbf{A}) = 0$ does not imply $\mathbf{A} = 0$)

b) For a general matrix $\mathbf{M}$, not necessarily symmetric, show that $\|\mathbf{M}\|_F \geq \rho(\mathbf{M})$ where $\|.\|_F$ is the Frobenius norm.