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Exercise 4

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Problem 1. *(Distribution of eigenvalues)* Use Gerschgorin's Theorem to find the smallest regions in which the eigenvalues of the matrix $\boldsymbol{A}$ are concentrated. Is $\boldsymbol{A}$ positive definite? Determine the smallest interval $[\lambda_{\min}, \lambda_{\max}]$ in which the real part of the eigenvalues are distributed.

$$\boldsymbol{A} = \begin{pmatrix} 10 & 0.1 & 1 & 0.9 & 0 \\ 0.2 & 9 & 0.2 & 0.2 & 0.2 \\ 0.3 & -0.1 & 5+i & 0 & 0.1 \\ 0 & 0.6 & 0.1 & 6 & -0.3 \\ 0.3 & -0.3 & 0.1 & 0 & 1 \end{pmatrix}$$

Gerschgorin's Theorem: Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, with entries a_{ij}, be given. For $i, j \in \{1, \dots, n\}$ let $R_i = \sum_{\substack{j=1 \\ j\neq i}}^n |a_{ij}|$ and $C_j = \sum_{\substack{i=1 \\ i\neq j}}^n |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries. Then every eigenvalue of $\boldsymbol{A}$ lies within at least one of the discs centered at a_{ii} with radius $\min\{R_i, C_i\}$.

Note that if one of the discs is disjoint from the others then it contains exactly one eigenvalue. If the union of m discs is disjoint from the union of the other $n-m$ discs then the former union contains exactly m and the latter $n-m$ eigenvalues of $\boldsymbol{A}$.

Problem 2. *(Distribution of eigenvalues)* Use Schur's inequality to find the region in which all eigenvalues of the matrix $\boldsymbol{A}$ are concentrated. Compare the obtained region with the solution by Gerschgorin.

$$\boldsymbol{A} = \begin{pmatrix} 10 & 0.1 & 1 & 0.9 & 0 \\ 0.2 & 9 & 0.2 & 0.2 & 0.2 \\ 0.3 & -0.1 & 5+i & 0 & 0.1 \\ 0 & 0.6 & 0.1 & 6 & -0.3 \\ 0.3 & -0.3 & 0.1 & 0 & 1 \end{pmatrix}$$

Schur's Inequality: Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, with entries a_{ij} and eigenvalues λ_i, be given. Then the inequality $\sum_{i=1}^n |\lambda_i|^2 \leq \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 = \|\boldsymbol{A}\|_2^2$ holds. Equality is attained if $\boldsymbol{A}$ is normal.

Problem 3. Prove the inequality $(1+\beta)(1+\frac{\gamma}{\beta}) \geq (1+\sqrt{\gamma})^2$.

Cauchy-Bunyakovsky-Schwarz Inequality: If $\boldsymbol{u} = (u_1, u_2, \dots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \dots, v_n)$ are two real vectors, then the inequality

$$\left(\sum_{i=1}^n u_i v_i\right)^2 \leq \sum_{i=1}^n u_i^2 \sum_{j=1}^n v_j^2 \qquad \Leftrightarrow \qquad \langle \boldsymbol{u}, \boldsymbol{v}\rangle \leq \langle \boldsymbol{u}, \boldsymbol{u}\rangle\langle \boldsymbol{v}, \boldsymbol{v}\rangle$$

holds. Equality is attained whenever $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent.

Problem 4. *(Spike model)* Fix $p = 500$ as the dimension of the space $\mathbb{R}^p$. Suppose that the data is generated from two one dimensional subspaces modeled by $\sqrt{0.2}G_1\boldsymbol{v}_1$ and $\sqrt{0.5}G_2\boldsymbol{v}_2$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal unit norm vectors in $\mathbb{R}^p$, and G_1 and G_2 are independent standard normal random variables. The high dimensional noise $\boldsymbol{U} \in \mathbb{R}^p$ is independent of both G_1 and G_2 and is modeled as a standard normal random vector. The covariance matrix of this model $\boldsymbol{X} = \boldsymbol{U} + \sqrt{0.2}G_1\boldsymbol{v}_1 + \sqrt{0.5}G_2\boldsymbol{v}_2$ is described by:

$$\mathrm{Cov}(\boldsymbol{X}) = \boldsymbol{I}_p + 0.2\boldsymbol{v}_1\boldsymbol{v}_1^{\mathrm{T}} + 0.5\boldsymbol{v}_2\boldsymbol{v}_2^{\mathrm{T}}.$$

Suppose that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are i.i.d. distributed with $\mathrm{Cov}(\boldsymbol{X}_i) = \mathrm{Cov}(\boldsymbol{X})$.

a) Find the minimum number n_2 of samples such that only the dominant eigenvalue is visible. Calculate the distance $\langle \boldsymbol{v}_2, \boldsymbol{v}_{\mathrm{dom}} \rangle$ for this case.

b) Find the minimum number n_1 of samples such that both dominant eigenvalues are visible. Calculate the distance $\langle \boldsymbol{v}_2, \boldsymbol{v}_{\mathrm{dom}} \rangle$ for this case. Sketch the Marchenko-Pastur density for the latter case along with both dominant eigenvalues of the sample covariance matrix $\mathbf{S}_n$.