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Exercise 1
- Proposed Solution -

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Solution of Problem 1

a) Since $\mathbf{M}$ is a non-negative definite matrix, it can be written as:

$$\mathbf{M} = \sum_i \lambda_i \mathbf{v}_i \mathbf{v}_i^T,$$

with $\lambda_i \geq 0$ and $\mathbf{v}_i$ its eigenvector. The eigenvectors $\mathbf{v}_i$'s for $i = 1, \ldots, n$ form an orthonormal basis for $\mathbb{R}^n$ and therefore each vector $\mathbf{x} \in \mathbb{R}^n$ can be written as:

$$\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{v}_i$$

for some $\alpha_i \in \mathbb{R}^n$. Using this expansion, it can be seen that:

$$\mathbf{M}\mathbf{x} = \sum_{i=1}^n \alpha_i \lambda_i \mathbf{v}_i,$$

and finally:

$$\mathbf{x}^T \mathbf{M}\mathbf{x} = \mathbf{x}^T \left(\sum_{i=1}^n \alpha_i \lambda_i \mathbf{v}_i\right) = \sum_{i=1}^n \lambda_i \alpha_i^2 \geq 0.$$

b) If $\mathbf{M}$ is positive definite, same proof applies here with the difference that λ_i's are strictly positive. In this case since $\mathbf{x} \neq 0$, at least one α_i is non-zero, say $\alpha_1 \neq 0$. Then $\alpha_1^2 \lambda_1 > 0$ which implies $\mathbf{x}^T \mathbf{M}\mathbf{x} > 0$.

Solution of Problem 2

a) Since $\mathbf{W} \succeq \mathbf{V}$, $\mathbf{W} - \mathbf{V}$ is non-negative definite. Therefore $\mathbf{x}^T(\mathbf{W} - \mathbf{V})\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, which means:

$$\mathbf{x}^T \mathbf{W}\mathbf{x} \geq \mathbf{x}^T \mathbf{V}\mathbf{x}.$$

Using Courant-Fischer theorem, it is known that:

$$\max_{S:\dim(S)=k} \ \min_{\mathbf{x}\in S; \|x\|_2=1} \mathbf{x}^T \mathbf{W}\mathbf{x} = \lambda_k(\mathbf{W}).$$

and

$$\max_{S:\dim(S)=k} \ \min_{x\in S; \|x\|_2=1} \mathbf{x}^T \mathbf{V}\mathbf{x} = \lambda_k(\mathbf{V}).$$

However $\mathbf{x}^T \mathbf{W}\mathbf{x} \geq \mathbf{x}^T \mathbf{V}\mathbf{x}$ implies that $\lambda_k(\mathbf{W}) \geq \lambda_k(\mathbf{V})$.

b) Since $\mathbf{W} \succeq \mathbf{V}$, $\mathbf{W} - \mathbf{V}$ is non-negative definite. Therefore $\mathbf{x}^T(\mathbf{W} - \mathbf{V})\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Choose $\mathbf{x} = \mathbf{e}_i$ where $\mathbf{e}_i$ is ith canonical basis with all zero elements except the ith element equal to one. Namely $\mathbf{e}_i(j) = 0$ for $j \neq i$ and $\mathbf{e}_i(i) = 1$. For example:

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Therefore $\mathbf{e}_i^T(\mathbf{W} - \mathbf{V})\mathbf{e}_i = w_{ii} - v_{ii}$ and since $\mathbf{W} - \mathbf{V} \succeq 0$, $w_{ii} - v_{ii} \geq 0$. $v_{ii} \leq w_{ii}$, for $i = 1, \ldots, n$

c) Similar to the previous problem, choose the vector $\mathbf{e}_{ij}$ such that $\mathbf{e}_{ij}(k) = 0$ for $j \neq i, j$ and $\mathbf{e}_{ij}(i) = 1$ and $\mathbf{e}_{ij}(j) = -1$. For example:

$$\mathbf{e}_{23} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix}$$

Since $\mathbf{W} - \mathbf{V} \succeq 0$, $\mathbf{e}_{ij}^T(\mathbf{W} - \mathbf{V})\mathbf{e}_{ij} \geq 0$, but:

$$(\mathbf{W} - \mathbf{V})\mathbf{e}_{ij} = \begin{bmatrix} (w_{1i} - v_{1i}) - (w_{1j} - v_{1j}) \\ (w_{2i} - v_{2i}) - (w_{2j} - v_{2j}) \\ \vdots \\ (w_{ni} - v_{ni}) - (w_{nj} - v_{nj}) \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{e}_{ij}^T(\mathbf{W} - \mathbf{V})\mathbf{e}_{ij} &= [(w_{ii} - v_{ii}) - (w_{ij} - v_{ij})] - [(w_{ji} - v_{ji}) - (w_{jj} - v_{jj})] \\ & [w_{ii} + w_{jj} - 2w_{ij}] - [v_{ii} + v_{jj} - 2v_{ij}] \,. \end{aligned}$$

Since $\mathbf{e}_{ij}^T(\mathbf{W} - \mathbf{V})\mathbf{e}_{ij} \geq 0$, it holds that: $v_{ii} + v_{jj} - 2v_{ij} \leq w_{ii} + w_{jj} - 2w_{ij}$.

d) From the second part of the exercise, $v_{ii} \leq w_{ii}$, for $i = 1, \ldots, n$. Therefore :

$$\mathrm{tr}(\mathbf{V}) = \sum_{i=1}^{n} v_{ii} \leq \sum_{i=1}^{n} w_{ii} = \mathrm{tr}(\mathbf{W}).$$

e) Note that $\det(\mathbf{V}) = \prod_{i=1}^n \lambda_i(\mathbf{V})$ and $\det(\mathbf{W}) = \prod_{i=1}^n \lambda_i(\mathbf{W})$. Using the first part of this exercise $\lambda_i(\mathbf{V}) \leq \lambda_i(\mathbf{W})$, for $i = 1, \ldots, n$. Since all eigenvalues are non-negative, it holds that $\det(\mathbf{V}) \leq \det(\mathbf{W})$.

Solution of Problem 3

a) Consider $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The eigenvalues are 0 which give $\rho(\mathbf{M}) = 0$. However $\mathbf{M} \neq 0$.

b) Let $\mathbf{x}$ be the eigenvector corresponding to the eigenvalue λ with $|\lambda| = \rho(\mathbf{M})$. Without loss of generality, it can be assumed that $\|\mathbf{x}\|_2 = 1$. Therefore:

$$\rho(\mathbf{M})^2 = |\lambda|^2 \|x\|^2 = \|\mathbf{Mx}\|_2^2 = \sum_i (\sum_j m_{ij} x_j)^2.$$

Using Cauchy-Schwartz inequality, we have:

$$(\sum_j m_{ij} x_j)^2 \leq (\sum_j m_{ij}^2)(\sum_j x_j^2)$$

But since $\|\mathbf{x}\|_2 = 1$, then:

$$\rho(\mathbf{M})^2 = \sum_i (\sum_j m_{ij} x_j)^2 \leq \sum_{i,j} m_{ij}^2 = \|\mathbf{M}\|_F^2.$$