Prof. Dr. Rudolf Mathar, Dr. Gholamreza Alirezaei, Dr. Arash Behboodi

Exercise 4
- Proposed Solution -

Friday, November 25, 2016

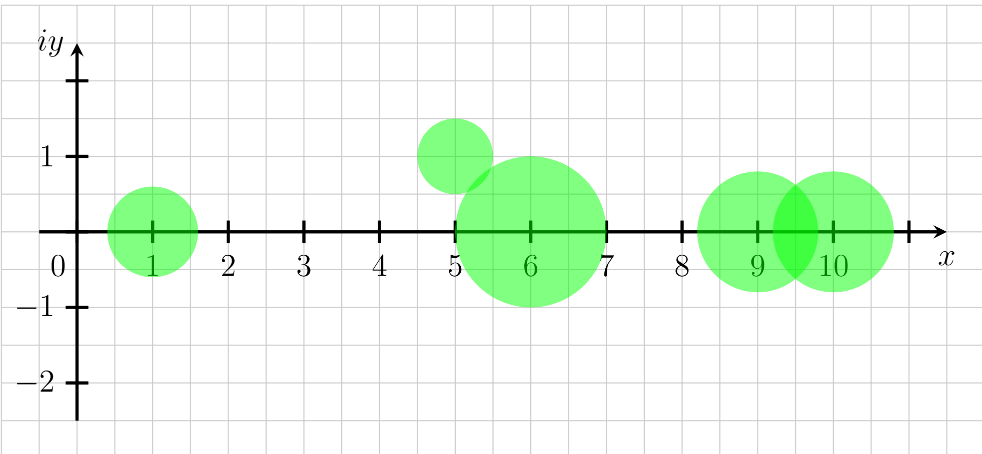
Solution of Problem 1

The radii $r_i = \min\{R_i, C_i\}$ of the discs are calculated by the aid of $R_i = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|$ and $C_j = \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}|$, and are given in the following table. The diagonal elements of $\boldsymbol{A}$ are the centers of the discs.

Table 1: The centers and radii of Gerschgorin's circles

i	a_{ii}	r_i	R_i	C_i
1	10	0.8	2.0	0.8
2	9	0.8	0.8	1.1
3	5+i	0.5	0.5	1.4
4	6	1.0	1.0	1.1
5	1	0.6	0.7	0.6

From the below figure we can observe that all areas of the circles are located on the right side of the plane. But having positive eigenvalues is not sufficient for $\boldsymbol{A}$ being positive definite. Since it is not symmetric, it will not be positive definite. Furthermore, we observe the limits $\lambda_{\min} = a_{55} - r_5 = 0.4$ and $\lambda_{\max} = a_{11} + r_1 = 10.8$. Note that since the disc located at a_{55} is disjoint from the others it contains exactly one of the eigenvalues.



Solution of Problem 2

Using Schur's inequality the region in which all eigenvalues of the matrix $\boldsymbol{A}$ are concentrated, is a circle at zero with radius given by $\|\boldsymbol{A}\|_2^2 = 246.74$. The calculation of $\|\boldsymbol{A}\|_2^2$ is usually a simple task and it only delivers a rough idea about the location of eigenvalues. The solution by Gerschgorin yields better results at the cost of computational complexity.

Solution of Problem 3

Note that both γ and β are positive. Rewrite $(1+\sqrt{\gamma})^2$ as $(1\cdot 1+\sqrt{\beta}\cdot\sqrt{\frac{\gamma}{\beta}})^2$ and apply the Cauchy-Bunyakovsky-Schwarz inequality to obtain $(1+\beta)(1+\frac{\gamma}{\beta}) \geq (1+\sqrt{\gamma})^2$. Equality holds, when $(1,\sqrt{\beta})$ and $\left(1,\sqrt{\frac{\gamma}{\beta}}\right)$ are linearly dependent, i.e., for $\beta=\gamma=1$.

Solution of Problem 4

a) The dominant eigenvalue λ_{dom} is visible when the ratio $\gamma_2 = \frac{p}{n_2}$ is less than β_{dom}^2. With $\beta_{\text{dom}} = \beta_2 = 0.5$ we obtain $n_{\min} = n_2 = \frac{p}{\beta_2^2} = 2000$. For this number of samples, the dominant eigenvalue of the sample covariance $\mathbf{S}_n$ tends to $(1+\sqrt{\gamma_2})^2 = (1+0.5)^2 = 2.25 \gg 1.5$. The distance $\langle \boldsymbol{v}_2, \boldsymbol{v}_{\text{dom}}\rangle = \frac{1-\gamma_1/\beta_1^2}{1-\gamma_1/\beta_1}$ is equal to zero. Figure 1 shows eigenvalue distributions for this choice.

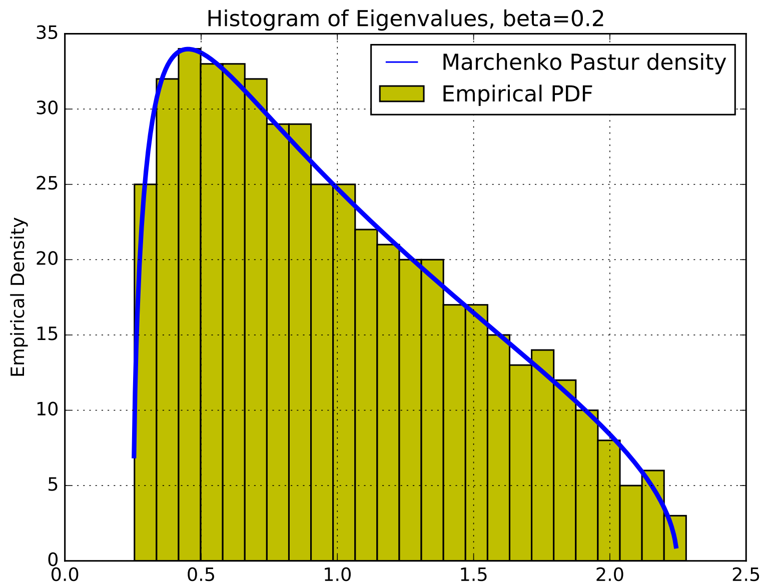


Figure 1: Eigenvalues of $\mathbf{S}_n$ for Spike model with $\beta_1 = 0.2, \beta_2 = 0.5$, $n = 2000$

b) To see both eigenvalues the ratio $\gamma_1 = \frac{p}{n_1}$ must be less than β_1^2. With $\beta_1 = 0.2$ we obtain $n_1 = \frac{p}{\beta_1^2} = 12500$. For this number of samples, the dominant eigenvalue λ_{dom} of the sample covariance $\mathbf{S}_n$ tends to $(1+\beta_2)(1+\frac{\gamma_1}{\beta_2}) = 1.5 \cdot 1.08 = 1.62 \approx 1.5 = 1+\beta_2$. The distance $\langle \boldsymbol{v}_2, \boldsymbol{v}_{\text{dom}}\rangle = \frac{1-\gamma_1/\beta_2^2}{1-\gamma_1/\beta_2}$ is equal to $0.913 \approx 1$ which shows that $\boldsymbol{v}_2$ is nearly a unit norm vector parallel to the dominant eigenvector $\boldsymbol{v}_{\text{dom}}$. Figure 2 shows eigenvalue distributions for this choice.

By enlarging n to 50000 both eigenvalues β_1 and β_2 become visible in the Marchenko-Pastur density as shown in Figure 3.

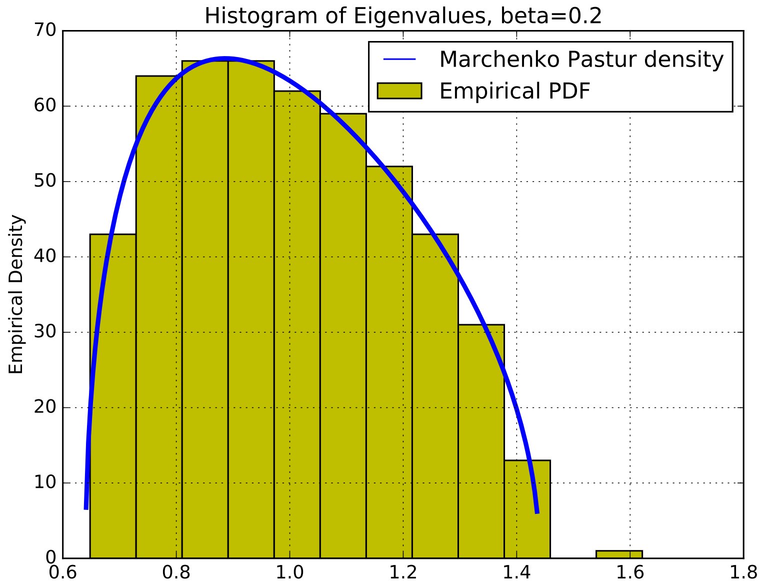


Figure 2: Eigenvalues of $\mathbf{S}_n$ for Spike model with $\beta_1 = 0.2, \beta_2 = 0.5$, $n = 12500$

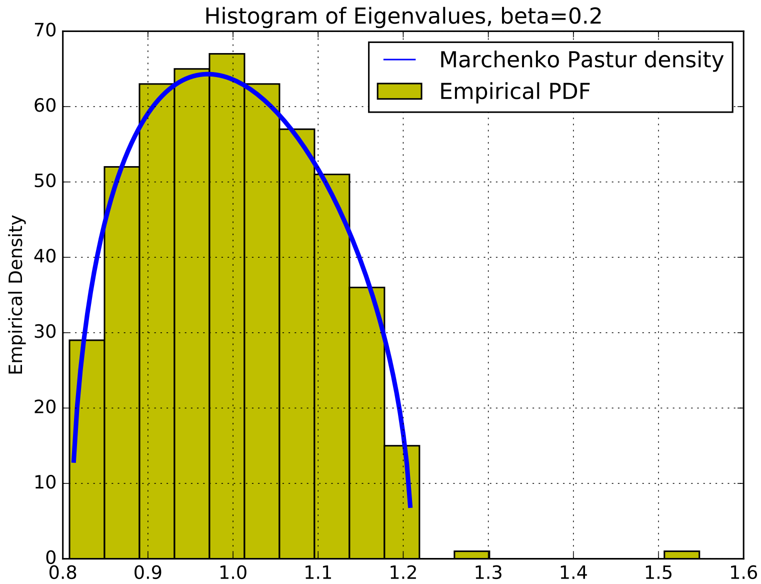


Figure 3: Eigenvalues of $\mathbf{S}_n$ for Spike model with $\beta_1 = 0.2, \beta_2 = 0.5$, $n = 50000$