$$
\begin{aligned}
I(X;Y) &= h(X) - h(X|Y) \\
&= -\int f(x)\log f(x)\,dx + \iint f(x,y)\log f(x|y)\,dx\,dy \\
&= \iint f(x,y)\log\frac{f(x|y)}{f(x)}\,dx\,dy \\
&= \iint f(x,y)\log\frac{f(x,y)}{f(x)f(y)}\,dx\,dy \qquad (*)
\end{aligned}
$$

also showing interchangeability of X and Y.

Def. 2.5.6. The relative entropy or Kulback-Leibler divergence between two densities f and g is defined as

$$D(f\|g) = \int f(x)\log\frac{f(x)}{g(x)}\,dx. \quad \lrcorner$$

From $(*)$ it follows that

$$I(X;Y) = D\big(f(x,y)\,\|\,f(x)\cdot f(y)\big) \qquad (**)$$

Th. 2.5.7. $D(f\|g) \geq 0$

with equality iff $f=g$ (almost everywhere, a.e.) $\lrcorner$

Proof. Let $S=\{x \mid f(x)>0\}$ the support of f. Then

$$
\begin{aligned}
-D(f\|g) &= \int_S f\log\frac{g}{f} \\
&\leq \log\int_S f\frac{g}{f} \qquad \text{(Jensen inequality, La 2.1.6)} \\
&\qquad\qquad\qquad\quad (f \text{ concave}: E f(X) \leq f(EX)) \\
&= \log\int_S g \leq \log 1 = 0
\end{aligned}
$$

Equality holds iff $f=g$ a.e. $\blacksquare$

Corollary 2.5.8.

a) $I(X;Y) \geq 0$ with equality iff X and Y are independent.

b) $h(X|Y) \leq h(X)$ with equality iff X, Y are independent.

c) $-\int f \log f \leq -\int f \log g$

Proof. a) follows from (**)

b) $I(X;Y) = h(X) - h(X|Y) \geq 0$ by a)

c) By Def. of $D(f\|g)$. ☒

Th. 2.5.9. (Chain rule for diff. entropy)

$$h(X_1, \dots, X_n) = \sum_{i=1}^{n} h(X_i \mid X_1, \dots, X_{i-1}).$$

Proof. From the definition it follows that

$$h(X, Y) = h(X) + h(Y|X)$$

This implies

$$h(X_1, \dots, X_i) = h(X_1, \dots, X_{i-1}) + h(X_i | X_1, \dots, X_{i-1}) \quad \forall i \in \mathbb{N}$$

The assertion follows by induction. ☒

Corollary 2.5.10.

$$h(X_1, \dots, X_n) \leq \sum_{i=1}^{n} h(X_i)$$

with equality iff $X_1, \dots, X_n$ are stoch. independent.

<u>Th. 2.5.11.</u> Let $X \in \mathbb{R}^n$ with density $f(x)$, $A \in \mathbb{R}^{n \times n}$ of full rank, $b \in \mathbb{R}^n$. Then

$$h(AX+b) = h(X) + \log|A| .$$

(Note that $|A| = |\det A|$)

<u>Proof.</u> If $X \sim f(x)$, then $Y = AX+b \sim \frac{1}{|A|} f(A^{-1}(y-b))$, $x, y \in \mathbb{R}^n$.

$$-\int \frac{1}{|A|} f(A^{-1}(y-b)) \log\left(\frac{1}{|A|} f(A^{-1}(y-b))\right) dy$$

$$= -\log\frac{1}{|A|} - \int \frac{1}{|A|} f(A^{-1}y) \log(f(A^{-1}y))\, dy$$

$$= -\log\frac{1}{|A|} - \int f(x) \log f(x)\, dx$$

$$= \log|A| + h(X). \qquad \blacksquare$$

<u>Th. 2.5.12.</u> Let $X \in \mathbb{R}^n$ abs. cont. with ~~f(x)~~ density $f(x)$ and $\mathrm{Cov}(X) = C$, C pos. definite. Then

$$h(X) \le \frac{1}{2} \ln\left((2\pi e)^n |C|\right),$$

i.e., $N_n(\mu, C)$ has largest entropy amongst all r.v. with pos. def. covariance matrix C. ⌟

<u>Proof.</u> W.l.o.g. assume $EX = 0$ (see Th. 2.5.11)

Let $\varphi(x) = \frac{1}{(2\pi)^{n/2} |C|^{1/2}} \exp\left\{-\frac{1}{2} x^T C^{-1} x\right\}$ density of $N_n(0, C)$.

Let $X \sim f(x)$, $EX = 0$, $\mathrm{Cov}(X) = E(XX^T) = \int x x^T f(x)\,dx$.

$$\begin{aligned}
h(X) &= -\int f(x) \ln f(x)\,dx \\
&\le -\int f(x) \ln \varphi(x)\,dx \quad (\text{Cor. } 2.5.8\ c)) \\
&= -\int f(x) \ln\left[\frac{1}{(2\pi)^{n/2}|C|^{1/2}} \exp\left\{-\tfrac{1}{2} x^T C^{-1} x\right\}\right] dx \\
&= -\ln\left[\frac{1}{(2\pi)^{n/2}|C|^{1/2}}\right] + \frac{1}{2}\int x^T C^{-1} x\, f(x)\,dx \\
&= \ln\left((2\pi)^{n/2}|C|^{1/2}\right) + \frac{1}{2}\int \mathrm{tr}\left(C^{-1} x x^T\right) f(x)\,dx \\
&= \cancel{\quad '' \quad + \tfrac{1}{2}\left[C^{-1}\int\right.} \\
&= \quad '' \quad + \frac{1}{2}\,\mathrm{tr}\, C^{-1} \underbrace{\int x x^T f(x)\,dx}_{= C} \\
&= \ln\left((2\pi)^{n/2}|C|^{1/2}\right) + \frac{n}{2} \\
&= \ln\left((2\pi e)^{n/2} |C|^{1/2}\right). \quad \blacksquare
\end{aligned}$$