An Analytical Foundation for Resource Management in Wireless Communication

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***Abstract*— The maximization of a ratio of the form f(x)/x, with f some "S-curve", plays a central role in several important problems involving resource management for data communication over a wireless medium. This includes decentralized power control, power and data rate assignment for maximal network throughput in a 3G-CDMA context, and power and coding rate choice for multi-media files which have been scalably encoded, as with the JPEG-2000 and MPEG-4 standards. In this note, the ratio f(x)/x, where f is a real-valued, univariate "s-shaped" function, is shown to be quasi-concave, and to always have a unique global maximizer, which can be identified graphically. The analysis is strictly based on geometrical properties derived from the sigmoidal shape, imposing no specific algebraic functional form ("equation") on the function. Hence, it applies to a wide range of practical situations.**

I. Introduction

Many radio resource optimizations of practical interest depend critically on the maximization of an expression of the form $f(x)/x$. The specific function and its argument depend on the problem being analyzed; but f is, typically, monotonically increasing. This maximization may be embedded into a larger optimization.

An example of interest arises when a transmitter with a limited supply of energy wishes to choose optimally its transmission power for data communication over a wireless medium in the presence of interference. Reference [7] discusses in detail why a ratio of the form $f(x)/x$ makes sense as an objective function for this situation. When several mutually-interfering terminals share a wireless channel, a decentralized power-allocation algorithm can be obtained by modeling each terminal as choosing its transmission power in order to maximize its own "utility function", of the form $f(x)/x$. Reference [3] provides an introductory discussion based on several previous works investigating energy management for wireless data applications, in which the maximization of a ratio of the form $f(x)/x$ plays a central role. This is also discussed in [7] for terminals with dissimilar transmission rates, as expected in 3G networks.

A related, but distinct inquiry is found in [8], a work relevant to a VSG-CDMA system, a technique part of 3G standards. This reference seeks centralized power and data rate allocations for many terminals in order to maximize the network *weighted* throughput. It shows that the first-order optimizing conditions require that some terminals operate with a signal-to-interference ratio (SIR) which maximizes $f(x)/x$.

In the applications above, the function f is a simple transformation of the "frame-success" function, f_s, which yields the probability of success of the transmission of a data packet, in terms, of the signal-to-interference ratio (SIR). This function depends on physical attributes of the system, including the binary modulation technique, the forward error detection scheme, the nature of the channel, and details of the receiver. Obtaining an exact expression for this function for a realistic model of a wireless communication setting may be prohibitively difficult or impossible. And even when this function is available, it may be intractable or highly inconvenient, and highly dependent on the chosen physical layer configuration. However, one can safely assume that, whatever this function is, its graph is always "sigmoidal" (S-shaped), as shown in fig. 1. Therefore, it is desirable to understand the behavior of the ratio $f(x)/x$, when *all that is known* about f is that its graph is S-shaped. This should lead to results appropriate for many interesting situations, regardless of modulation techniques and other physical layer arrangements.

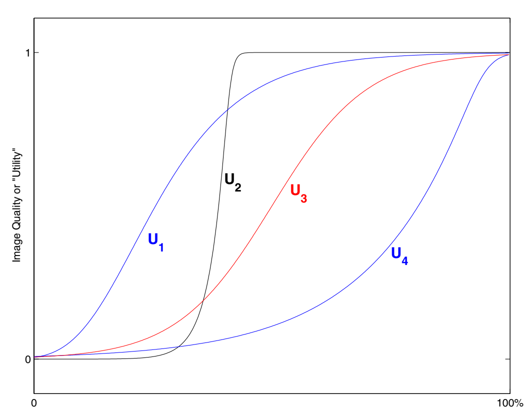

Fig. 1. Some S-curves

In other applications, the function in the numerator may have a completely different meaning. For instance, [6] focuses on an energy-limited terminal with a long stream of files to transmit over a wireless channel. The files correspond to scalably encoded images (as with the JPEG 2000 standard), which can be truncated at an arbitrary point and decoded. Two key variables are jointly optimized: transmission power, and the number of bits of each file to be decoded. In the analysis, a ratio of the form $g(y)/y$ arises, with g yielding a measure of the quality of a decoded image, as a function of the number of bits decoded, y. The function g is also assumed to be sigmoidal, which is consistent with certain psychophysical experiments.

There are additional practical reasons why the S shape may be chosen for modeling a monotonic function of interest. An arbitrary S-curve starts out convex and smoothly transitions to concave. But the inflexion (transition) point is arbitrarily placed. Therefore, this curve in fact contains as special cases a "mostly" concave curve (inflexion point is "very close" to the origin) and a "mostly" convex curve (inflexion point is "very far" from the origin). Furthermore, the "ramp" of the S-curve may be nearly vertical, in which case the S-curve behaves like a "step" (threshold) function. Or this "ramp" can approximate a straight line, in which case the S-curve expresses a near linear relation. These shapes, shown in fig. 1, accommodate many situations of interest.

Problems involving the optimization of ratios of functions have been intensively studied in the last few decades, and are commonly called "fractional programming". These problems arise naturally in many contexts, including macroeconomics, finance, inventory control, and numerical analysis, among others. References [2], [10] are very recent surveys of this literature. However, the most general formulations studied in this literature involve ratios of concave and convex functions. In a few cases, the definitions of concavity and/or convexity are relaxed to include a somewhat larger class of functions. But, the sigmoidal functions studied herein are, by definition, neither concave nor convex (very loosely speaking they are "half and half"), and are, therefore, excluded from the current fractional programming literature.

This work investigates the maximization of the ratio $f(x)/x$ for any function f having the specified sigmoidal shape. "Sigmoidness" is captured in a strictly geometric manner, by assuming that the considered function "starts out" convex at the origin, and "smoothly" transitions to concave as it approaches a horizontal asymptote. The optimal solution is characterized strictly in terms of geometrical properties derived from this shape. Without imposing any particular algebraic functional form ("equation") on the considered functions, this note shows that the solution to this maximization problem always exists, is unique, and can be graphically described and determined. Additionally, the ratio $f(x)/x$ is shown to be quasi-concave.

Below, the considered class of functions is formally characterized. Then, the solution to the maximization problem of interest is derived. Subsequently, the quasi-concavity of the ratio is established. Finally, some closing comments are given. An appendix provides and develops certain key technical results.

II. Formalization of the functions of interest

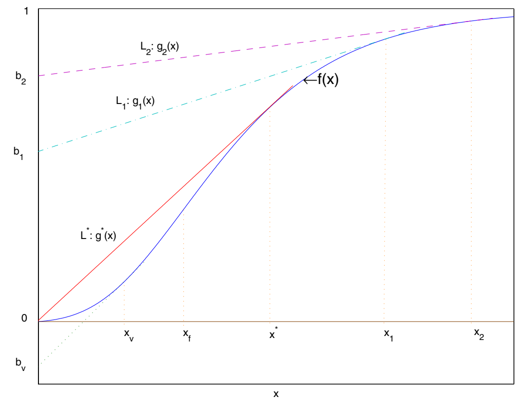

Fig. 2. A representative function and some of its tangents

A. Basic Assumptions

Figure (2) provides a graphical illustration of a function representative of the class of functions to be considered. Any such function, f, has the following characteristics:

1) Its domain is the non-negative part of the real line; that is, the interval $[0, \infty)$
2) Its range is the interval $[0, B)$, where, for convenience, and without loss of generality, we take $B = 1$.
3) It is increasing.
4) ("Initial convexity") It is strictly convex over the interval $[0, x_f]$, with x_f a positive number.
5) ("Eventual concavity") It is strictly concave over any interval of the form $[x_f, L]$, where L is a positive number greater than x_f
6) It has a continuous derivative.

Notice that *no* assumptions about the second derivative of the function f are explicitly made.

B. Immediately Implied Characteristics

1) Assumptions (1), (2) and (3) imply that $f(0) = 0$.
2) Assumptions (4) ("initial convexity") and (5) ("eventual concavity") imply that the function is continuous for any $x > 0$. (See Theorem 1.3, Chapter III, in reference [1]). And this implication, together with the preceding one further imply that f is continuous overall.
3) The "initial convexity" assumption (4) and the continuous derivative assumption (6) together imply that $f'(0) < \infty$ (See subsections (B.1) and (III-B.1)). This ensures that $\lim_{x \to 0} f(x)/x$ is finite, by L'Hopital rule
4) Assumption (6) also implies the continuity of f .

III. Maximization

Below, the following optimization problem is solved:

$$\text{Max: } f(x)/x \text{ subject to } 0 \leq x \leq M$$

A. An interior solution

First, it is presumed that a "stationary" point exists within the allowable range of x.

1) First-order conditions for a maximum: The first-order necessary conditions are:

$$f(x) - xf'(x) = 0 \tag{1}$$

It will prove useful to observe that the equation of a straight line tangent, at the point $(x_1, f(x_1))$, to the curve described by the graph of the function f can be written as

$$g_1(x) = f(x_1) + f'(x_1)(x - x_1) \text{ or } g_1(x) = b(x_1) + f'(x_1)x \tag{2}$$

where $b(x) := f(x) - xf'(x)$ represents the ordinate at the origin (y-intercept) of the straight line tangent at the point $(x, f(x))$ to the curve described by the graph of f (see fig. (2)). Therefore, equation (1) can be stated as $b(x) = 0$, which is discussed further in section (III-A.5).

2) Existence of a Solution: A solution to equation (1) always exists. This follows from these facts:

i) $b(x) = f(x) - xf'(x)$ is a continuous function.
ii) For sufficiently large x_L, $b(x_L) > 0$
iii) For any x_v in $(0, x_f]$, $b(x_v) < 0$.

Statement (i) follows directly from the fact that both $f(x)$ and $f'(x)$ have been assumed to be continuous.

Statement (ii) is a direct consequence of the fact that, by assumption, $\lim_{x\to\infty} f(x) = 1$. Hence, in the limit, the tangent line to the graph of f is the line $y = 1$. The y-intercept of this line is, of course, 1. So, $\lim_{x\to\infty} b(x) = 1$, for which $b(x)$ is bound to take on positive values "sooner or later".

Statement (iii) follows from the essential property of tangent lines of continuously differentiable strictly convex functions (see section (B.1)). Over the interval $[0, x_f]$, f is assumed to be strictly convex. Taking $x_2 = 0$ and x_1 equal to an arbitrary number in $(0, x_f]$, denoted as x_v, inequality (7) yields $f(0) > f(x_v) + f'(x_v) \cdot (0 - x_v)$ or, equivalently, $b(x_v) = f(x_v) - x_v f'(x_v) < 0$.

Statements (i), (ii), and (iii) above have been shown to be valid. These three facts imply the existence of an x^*satisfying $b(x^*) = 0$, because a continuous function cannot go from a negative to a positive value without taking on the value zero.

Furthermore, notice that the validity of statement (iii) immediately implies that any such x^*must be greater than x_f (that is, any such x^*must be in the interval over which f is concave), since, $x < x_f \rightarrow b(x) < 0$.

3) Uniqueness of the solution: In subsection (III-A.2) it was established that any solution to $b(x) = f(x) - xf'(x) = 0$ must must lie inside the interval where f is strictly concave. The uniqueness of this solution follows directly from the "monotone intercepts" corollary, presented in subsection (B.2). This results indicates that if x_1 and x_2 are points in an interval of the real line over which the function f is strictly concave, then $x_2 > x_1$implies that $b(x_2) > b(x_1)$. Hence, if $x^*$is such that $b(x^*) = 0$, any $x \neq x^*$must be such that $b(x) \neq 0$

4) Optimality of the solution: The derivative of the ratio $f(x)/x$ can be expressed as

$$\frac{xf'(x) - f(x)}{x^2} = -\frac{b(x)}{x^2} \tag{3}$$

with $b(x)$ as previously defined. The derivative is well-defined with the possible exception of the boundary value $x = 0$. The case $x = 0$ is discussed in the subsection (III-B). For the purposes of this section, x is assumed to be positive.

The monotone intercepts corollary of subsection (B.2) specifies that for any $x > x^*$, $b(x) > b(x^*) = 0$. Therefore, the ratio $f(x)/x$ is strictly *decreasing* for any $x > x^*$.

The same argument leads to the conclusion that the ratio $f(x)/x$ is strictly *increasing* for any $x_f < x < x^*$.

In subsection (III-A.2) it was established that $b(x) < 0$ for any x in $(0, x_f]$. Therefore, the derivative of the ratio $f(x)/x$ is positive for any such x, (see equation (3) above), which means this ratio is increasing over $(0, x_f]$.

In conclusion, the ratio $f(x)/x$ is less than $f(x^*)/x^*$ for any positive $x \neq x^*$.

5) Description of the solution: The characteristic tangent: The solution to the first-order necessary optimizing conditions given by equation (1) can be directly identified in the graph of the function f. Only one positive value, x^*, satisfies equation (1). $(x^*, f(x^*))$ is the only point at which a line tangent to the curve describing the function passes through the origin. Thus, the equation of any such tangent line is $g^*(x) = f'(x^*)x$. (See the tangent line drawn at x^* in fig. (2)). This tangent line is termed "the *characteristic tangent"* of a given sigmoidal function. Of course, different sigmoids may have the same characteristic tangent.

The value of the objective function at the solution, x^*, can be obtained graphically as the slope of the characteristic tangent, which is $f(x^*)/x^*$. This observation can be useful for conceptual "sensitivity analyses". The effect on the optimal solution of changing one sigmoid for another (for example via a change in certain parameter) immediately manifests itself, visually, through the new characteristic tangent, and its slope.

B. "Boundary" solution

The development so far has ignored the constraint that $x \leq M$ for some M. Below, this issue is addressed. Before that, the possibility that the optimal value be zero is formally discarded.

1) The non-optimality of x=0: By construction, and the application of L'Hopital rule, $\lim_{x\to 0} f(x)/x = f'(0) < \infty$. In sub-sections (III-A.2) and (III-A.4) it was discussed why the ratio $f(x)/x$ is increasing over the interval $(0, x_f]$. Hence, $x = 0$ is *not* the maximizer.

2) The global optimality of the smallest of M and x^*: Given the discussion in subsections (III-A.4) and (III-B.1), it is clear that the ratio $f(x)/x$ is increasing over the interval $[0, x^*]$, where x^* is the only value of x satisfying the first-order necessary optimizing conditions given by equation (1). Hence, if the maximum allowable value for x, denoted as M, is less than x^*, $f(M)/M$ is the highest achievable value for the ratio $f(x)/x$. But if x^* is less than M, $x = x^*$ is clearly the optimizing choice. Therefore, the smallest of the numbers M and x^* is the global maximizer.

IV. The Quasi-concavity of $f(x)/x$

In the preceding development, it has been determined that, for the class of functions under consideration (see section(II-

A)), the ratio $f(x)/x$ is "single-peaked"; that is, there is a number x^* such that this ratio is strictly increasing for all $x \in [0, x^*)$ and strictly decreasing for all $x \in (x^*, \infty)$. This implies the quasi-concavity of this ratio. For a general discussion about quasi-concavity and various related concepts and results, see [5].

Below, the definition of quasi-concavity is given, and the compliance of $f(x)/x$ with this definition is formally established.

A. Definition of Quasi-concavity

Definition: The function $h : I \to R$, defined on an interval $I \subset \Re$, is said to be quasi-concave if its upper contour sets, $\{x \in I \,:\, h(x) \geq t\}$, are convex sets; that is, for any $t \in \Re$, any $\alpha \in [0, 1]$, and any $x_1, x_2 \in I$, $h(x_1) \geq t$ and $h(x_2) \geq t$ imply that

$$h(\alpha x_1 + (1 - \alpha)x_2) \geq t \quad (4)$$

The function h is said to be *strictly* quasi-concave if the implied inequality in (4) holds strictly whenever $x_1 \neq x_2$ and $\alpha \in (0, 1)$.

B. Verification of Quasi-concavity

The function $f(x)/x$ is strictly quasi-concave.

Proof:

For notational convenience, let $h(x) \doteq f(x)/x$ and let $h(x^*) \doteq P^*$.

Let $t \in (0, P^*)$. Notice that verifying (4) is trivial for t outside this interval.

Suppose $0 \leq x_1 < x_2$, $h(x_1) \geq t$ and $h(x_2) \geq t$

Because $h(x)$ is continuous and strictly *increasing* in the interval $[0, x^*)$, there is an x'_t such that $h(x) \geq t$ for all x between x'_t and x^*, and $h(x) < t$ for $x < x'_t$. Likewise, since $h(x)$ is continuous and strictly *decreasing* in the interval (x^*, ∞), there is an x''_t such that $h(x) \geq t$ for all x between $x^*$and x''_t, and $h(x) < t$ for $x > x''_t$.

Then, clearly, any x for which $h(x) \geq t$ must be between x'_t and x''_t, and any x between x'_t and x''_t is such that $h(x) \geq t$. That is, $x'_t \leq x \leq x''_t \Leftrightarrow h(x) \geq t$.

Therefore, $h(x_1) \geq t$ and $h(x_2) \geq t$ implies $x'_t \leq x_1 < x_2 \leq x''_t$

And for $\alpha \in (0, 1)$, $x_1 < \alpha x_1 + (1 - \alpha)x_2 < x_2$. This implies $x'_t < \alpha x_1 + (1 - \alpha)x_2 < x''_t$, which further implies $h(\alpha x_1 + (1 - \alpha)x_2) \geq t$

Q.E.D.

V. Concluding remarks

The maximization of the ratio $f(x)/x$ for any function f having a "sigmoidal" shape has been studied, and its optimal solution been characterized without assuming any particular algebraic functional form ("equation") on the considered functions. "Sigmoidness" has been captured in a strictly geometric manner, by assuming that the considered functions "start out" convex at the origin, and "smoothly" transition to concave as they approach a horizontal asymptote. This *geometric* construction had not been found in the scientific literature, although sigmoidal functions have been studied in numerous contexts, including in technological, biological and socio-economic environments. On the basis of geometrical properties derived from this shape, this note shows that the solution to the maximization problem of interest always exists, is unique, and can be graphically described and determined. The graphical identification of the solution could be valuable as a conceptual tool to understand the meaning of the solution, as well as a "sensitivity analysis" tool, to visualize how a change in the considered function can impact the optimal solution. Central to the development and fully developed herein, the observation that the "y-intercepts" of concave and convex functions are monotonic may be useful beyond the particular aims of this note. Along the way, the ratio $f(x)/x$ has been shown to be quasi-concave, which is by no means obvious given the arbitrary sigmoidal shape of the function in the numerator. This fact can be beneficial in situations in which this maximization is embedded into a larger problem, as in the "game" discussed in references [3], [7], where certain important theorems and results (such as Debreu's "general equilibrium") can be invoked because of the quasi-concavity of this ratio.

The maximization of a ratio of the form $f(x)/x$, with f some "S-curve", is particularly relevant to several important problems involving resource management for data communication over a wireless medium. This includes decentralized power control, [3], [7], power and data rate assignment for maximal network throughput in a 3G-CDMA context, [8], and resource management for scalably-encoded visual information, as with the JPEG-2000 and MPEG-4 standards,[6].

Acknowledgment

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References

[1] Berkovitz, L.D. "Convexity and optimization in R^n", New York : J. Wiley, 2002

[2] Frenk, H. and S. Schaible. "Fractional Programming: Introduction and Applications", Encyclopedia of Optimization, eds. Floudas, C.A. and P.M. Pardalos, Dordrecht-Boston-London : Kluwer, 2002

[3] Goodman, D.J., N. Mandayam, "Power Control for Wireless Data", IEEE Personal Communications , Vol. 7, No. 2 , pp. 48 -54, April 2000

[4] Mas-Colell, A., M.D. Whinston and J.R. Green, "Microeconomic Theory", New York: Oxford Univ. Press, 1995

[5] Ponstein, J. "Seven Kinds of Convexity", SIAM Review, Vol. 9, No. 1, pp. 115-119, January 1967

[6] Rodriguez, V., "Resource Management for Scalably Encoded Information: The Case of Image Transmission Over Wireless Networks", IEEE ICME, Vol. 1, pp. 813 -816, 2003

[7] Rodriguez, V., "Robust Modeling and Analysis for Wireless Data Resource Management", IEEE WCNC, Vol. 2, pp. 717 -722, 2003

[8] Rodriguez, V., and D.J. Goodman, "Power and Data Rate Assignment for Maximal Weighted Throughput in 3G CDMA", IEEE WCNC, Vol.1, pp. 525-31, 2003

[9] Rudin, W., "Principles of Mathematical Analysis" (3rd ed), New York : McGraw-Hill, 1976

[10] Schaible, S. "Fractional Programming - a Recent Survey", Journal of Statistics and Management Systems, forthcoming, 2002

APPENDIX

Much of this as well as other relevant material can be found in reference [1], in particular in chapter III. The presentation here follows that in the mathematical appendix of reference [4]. However, the material of subsection (B.2) is not found in those references, and is developed in full here.

A. *Concave and convex functions*

Consider a function $f : I \rightarrow R$, defined on an interval $I \subset \Re$.

Definition: The function f is said to be concave if, $\forall x_1, x_2 \in I$ and $\alpha \in (0,1)$,

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha) f(x_2) \qquad (5)$$

The function f is said to be *strictly* concave if the above inequality holds strictly whenever $x_1 \neq x_2$.

Definition: The function f is said to be (strictly) *convex* if the function $-f$ is (strictly) concave.

B. *Properties of continuously differentiable concave and convex functions*

1) Tangent line Theorem: The continuously differentiable function $f : I \rightarrow R$, defined on an interval $I \subset \Re$, is concave if and only if, $\forall x_1, x_2 \in I$,

$$f(x_2) \leq f(x_1) + f'(x_1) \cdot (x_2 - x_1) \qquad (6)$$

This function is *strictly* concave if and only if the above inequality holds strictly $\forall (x_1 \neq x_2) \in I$.

The function f is convex if and only if, $\forall x_1, x_2 \in I$,

$$f(x_2) \geq f(x_1) + f'(x_1) \cdot (x_2 - x_1) \qquad (7)$$

This function is *strictly* convex if and only if the above inequality holds strictly $\forall (x_1 \neq x_2) \in I$.

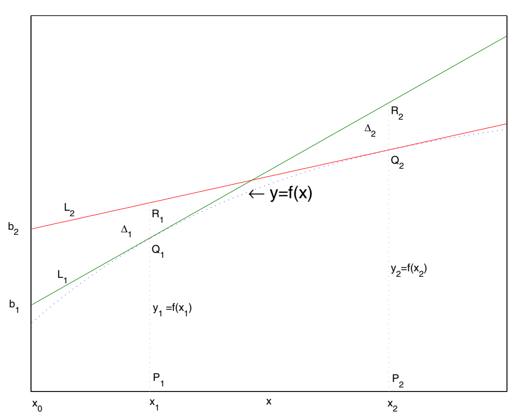

Fig. 3. Increasing Y intercepts

2) The Monotonicity of y-intercepts: Corollary: Let $f : I \rightarrow R$ denote a continuously differentiable *concave* function, defined on an interval $I \subset \Re$. Let x_0, x_1, x_2 be elements of I such that $x_0 < x_1 < x_2$. Then,

$$f(x_2) + (x_0 - x_2)f'(x_2) \geq f(x_1) + (x_0 - x_1)f'(x_1) \qquad (8)$$

If f is *strictly* concave the above inequality holds strictly.

Proof:

See figure (3). In this development, $i \in \{1, 2\}$.

First notice that $g_i(x) = f(x_i) + f'(x_i)(x - x_i)$ denotes the equation of a line tangent at the point (x_i, y_i) $(y_i \doteq f(x_i))$ to the the curve describing the graph of f.

Let $b_i \doteq f(x_i) + (x_0 - x_i)f'(x_i)$.

Thus, b_i is the "height" of tangent line L_i at the abscissa x_0, or its "intercept" with a vertical line drawn at x_0. Hence, inequality (8) can be restated as $b_2 > b_1$. In the special case $x_0 = 0$, b_i become the "y-intercept" or ordinate at the origin of the line L_i.

Let $\Delta_1 \doteq g_2(x_1) - y_1$ and $\Delta_2 \doteq g_1(x_2) - y_2$.

Geometrically, Δ_1 is the length of the segment Q_1R_1, which equals the difference between the "height" of the tangent L_2 and the value of the function f, both measured at the abscissa x_1. Δ_2 has an analogous interpretation.

Observe that the points (x_0, b_1), Q_1 and R_2 are all in the line L_1.

Likewise, (x_0, b_2), R_1 and Q_2 are all in the line L_2.

Therefore:

$$\frac{y_1 - b_1}{x_1 - x_0} = \frac{y_2 + \Delta_2 - b_1}{x_2 - x_0} \Rightarrow$$

$$b_1 = \frac{(x_2 - x_0)y_1 - (x_1 - x_0)(y_2 + \Delta_2)}{x_2 - x_1} \qquad (9)$$

$$\frac{y_2 - b_2}{x_2 - x_0} = \frac{y_1 + \Delta_1 - b_2}{x_1 - x_0} \Rightarrow$$

$$b_2 = \frac{(x_2 - x_0)(y_1 + \Delta_1) - (x_1 - x_0)y_2}{x_2 - x_1} \qquad (10)$$

Consequently:

$$b_2 - b_1 = \frac{(x_2 - x_0)\Delta_1 + (x_1 - x_0)\Delta_2}{x_2 - x_1} \qquad (11)$$

By construction, $x_0 < x_1 < x_2$.

By inequality (6), both Δ_1 and Δ_2 are non-negative, and both are positive if f is strictly concave.

Therefore, the right hand side of equation (11) is non-negative, and it is positive, if f is strictly concave.

That is, if f is concave, $b_2 \geq b_1$, and $b_2 > b_1$if f is strictly concave.

Q.E.D.

Given the fact that $-f$ is concave whenever f is convex (see section(A)), the following result is immediate:

Corollary: Let $f : I \rightarrow R$ denote a continuously differentiable *convex* function, defined on an interval $I \subset \Re$. Let x_0, x_1, x_2 be elements of I such that $x_0 < x_1 < x_2$. Then,

$$f(x_2) + (x_0 - x_2)f'(x_2) \leq f(x_1) + (x_0 - x_1)f'(x_1) \qquad (12)$$

If f is *strictly* convex the above inequality holds strictly.